Toric varieties from cyclic matrix groups

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MAX-PLANCK-GESELLSCHAFT

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The motivation

- Polynomial equations provide the strongest algebraic invariants.
- Membership problem: is $x \in X$? If $x \notin \overline{X}$, then $x \notin X$.
- Dynamical systems ($=$ automata, affine programs).
  Theorem: $\overline{X}$ is computable.
  - $X \subseteq \mathrm{GL}_n(\mathbb{C})$: Derksen-Jeandel-Koiran (2005), Kauers-Zimmermann (2008)
  - general: Hrushovski-Ouaknine-Pouly-Worrel (2018)

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We want to compute the algebraic closure of X in $\mathbb{C}^4$ where

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We compute some powers:

$$\begin{aligned} M^0 &= \mathrm{diag}(1, 1), & M^1 &= \mathrm{diag}(-1, 1/5), \\ M^2 &= \mathrm{diag}(1, 1/25), & M^3 &= \mathrm{diag}(-1, 1/(125)), \\ M^{-2} &= \mathrm{diag}(1, 25), & M^{-1} &= \mathrm{diag}(-1, 5). \end{aligned}$$

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Rewrite as:

$$X = \underbrace{\left\{ \mathrm{diag}(1, 5^{2k}) \mid k \in \mathbb{Z} \right\}}_{X_1} \cup \underbrace{\left\{ \mathrm{diag}(-1, 5^{2k+1}) \mid k \in \mathbb{Z} \right\}}_{X_{-1}}.$$

Recall:

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Then we get:

$$\overline{X} = \overline{X_1} \cup \overline{X_{-1}} = \left\{\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mid x_{12} = x_{21} = 0, x_{11} = \pm 1, x_{22} \in \mathbb{C}\right\}$$

Warming up 〉 An example

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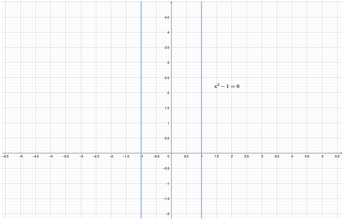
where the defining ideal is:

$$\begin{aligned} I &= (x_{11}^2 - 1, x_{12}, x_{21}) \\ &\subseteq \mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}] \end{aligned}$$



- $M \in \mathrm{Mat}_n(\mathbb{C})$
- $G = \langle \lambda \in \mathbb{C}^* \mid \ker(M - \lambda \,\mathrm{Id}_n) \neq 0 \rangle \cong G_{\mathrm{tor}} \oplus \mathbb{Z}^{\mathrm{rk}(G)}$
- $X_{>0}$ - semigroup generated by M

$$X_{>0} = \left\{ M^k \mid k \text{ positive integer} \right\}$$

- X - group generated by $M \in \mathrm{GL}_n(\mathbb{C})$

$$X = \langle M \rangle = \left\{ M^k \mid k \text{ integer} \right\}$$

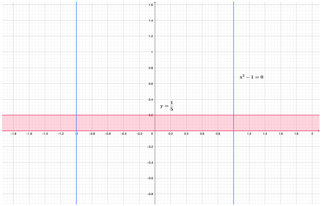
- $\overline{S}$ - Zariski closure of $S \subseteq \mathrm{Mat}_n(\mathbb{C})$ in $\mathbb{C}^{n^2}$

Our goal is to determine algebro-geometric properties of $\overline{X}$ or $\overline{X_{>0}}$.

Lemma

If M is invertible, then $\overline{X} = \overline{X_{>0}}$.

<u>Recall:</u> our warming-up example with $M = \mathrm{diag}(-1, 1/5)$.



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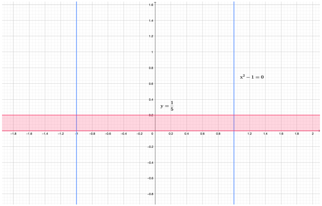

For M invertible, write $M \sim M_s M_u$ with M_s diagonal and M_u unipotent (Jordan normal form). Define then

$$X_s = \overline{\{M_s^k \mid k \in \mathbb{Z}\}} \text{ and } X_u = \overline{\{M_u^k \mid k \in \mathbb{Z}\}}$$

Lemma

If M is invertible, then $\dim \overline{X} = \dim X_s + \dim X_u$ and $\mathrm{irr}(\overline{X}) = \mathrm{irr}(X_s)$.

Theorem

Assume $M \in \mathrm{GL}_n(\mathbb{C})$*. Then* $\mathrm{irr}(\overline{X}) = |G_{\mathrm{tor}}|$ *and each irreducible component of* $\overline{X}$ *is a toric variety of dimension*

$$\dim \overline{X} = \begin{cases} \mathrm{rk}(G) & \text{if } M \text{ is diagonalizable,} \\ \mathrm{rk}(G)+1 & \text{otherwise.} \end{cases}$$

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Example

Let us look again at the matrix

$$M = \begin{pmatrix} -1 & 0 \\ 0 & 1/5 \end{pmatrix}.$$

Then M is diagonalizable and $G = \langle -1, 1/5 \rangle \cong \mathbb{Z}/(2) \oplus \mathbb{Z}$. We recover our two lines in $\mathbb{C}^4$.

Idea of proof

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   If $M = \mathrm{diag}(3, 1/3, 9)$, then $G = \langle 3, 1/3, 9 \rangle = \langle 3 \rangle \cong \mathbb{Z}$.

   - Get matrix of exponents: $A = (1\ \ -1\ 2) \in \mathrm{Mat}_{\mathrm{rk}(G)\times n}(\mathbb{C})$.
   - Compute kernel: $\ker_{\mathbb{Z}}(A) = \mathbb{Z}(1,1,0) \oplus \mathbb{Z}(-2,0,1)$.
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   For $q = |G_{\text{tor}}|$ define $X_i = \left\{M^{kq+i} \mid k \in \mathbb{Z}\right\}$. Then

   $$\overline{X} = \overline{X_0} \cup \ldots \cup \overline{X_{q-1}}.$$

   Conclude by observing

   $$X_0 \longleftrightarrow M^q \longleftrightarrow G^q = \langle x^q \mid x \in G\rangle \text{ torsion-free.}$$

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   Consider the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow M^\alpha = \begin{pmatrix} 1 & \alpha & \alpha(\alpha-1)/2 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$$

   satisfying $x_{ii} = 1, x_{12} = x_{23}, x_{i>j} = 0$.

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4. Conclude using lemmas.

□

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Example

Let d be a non-negative integer and define

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2^d \end{pmatrix} \in \mathbb{C}^2.$$

Then $\overline{X}$ is a smooth irreducible affine curve of degree d and equation $y = x^d$. It is rational and, when $d \geq 3$, singular at infinity.

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Example

Let $M \in \mathrm{Mat}_n(\mathbb{C})$ *be defined by*

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \rightsquigarrow M^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \mathrm{diag}(0,0,4).$$

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Then $X_{>0} = \{M\} \ \dot{\cup} \ \{\mathrm{diag}(0,0,2^k) \mid k \geq 2\}.$

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Then $X_{>0} = \{M\} \ \dot{\cup} \ \{\mathrm{diag}(0,0,2^k) \mid k \geq 2\}$. *Since the closure of* $\{\mathrm{diag}(0,0,2^k) \mid k \geq 2\}$ *has dimension* 1, *we get that*

$$\overline{X_{>0}} = \{M\} \ \dot{\cup} \ \{\mathrm{diag}(0,0,z) \mid z \in \mathbb{C}\}$$

and $\overline{X_{>0}}$ *is the disjoint union of a point and a line.*

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- In some cases, our results suffice also to deal with X non-cyclic. In general, we derive lower bounds.

Example

Define $X = \langle A, B \rangle$ where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then X contains as a subgroup $Y = \left\{ \mathrm{diag}(2^h, 3^h) \mid h \in \mathbb{Z} \right\}$ and so $\overline{X}$ is a plane in $\mathbb{C}^4$.

- Paul Görlach has shown in his PhD thesis (2020+) that, if X is generated by diagonal matrices, then the irreducible components of $\overline{X}$ are toric. Always true if X abelian?
- Work (2020+) of Majumandar, Ouaknine, Pouly, Worrel shows that toric varieties pop up as invariants also in the conext of linear hybrid automata. Properties?

- What happens if X is non-abelian?
- What happens over arbitrary fields?


thank you