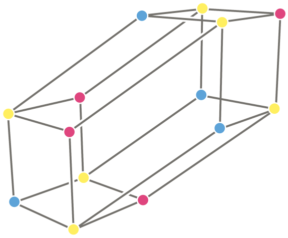
Vertices of balls: orders, codes, and tropical geometry

Mima Stanojkovski,
joint with Y. El Maazouz, M. Hahn, G. Nebe, B. Sturmfels



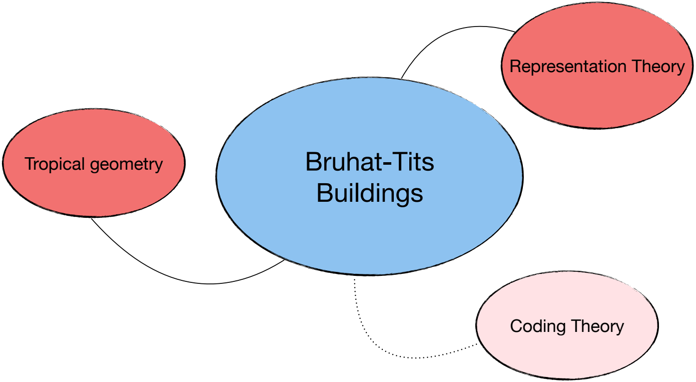
TR 195 SYMBOLIC TOOLS

Max-Planck-Institut für Mathematik in den Naturwissenschaften

RWTH AACHEN UNIVERSITY

Nonlinear Algebra Seminar - 27th April 2022

The project ⟩ Arithmetic and convexity in buildings



Main Goal: study orders via the collection of their stable lattices.
Related: spherical codes in buildings.

Let K be a field with a surjective valuation map

$$\mathrm{val} : K \to \mathbb{Z} \cup \{\infty\}.$$

Denote

- $\mathcal{O}_K = \{x \in K : \mathrm{val}(x) \geq 0\}$ is the valuation ring of K,
- $\mathfrak{m}_K = \{x \in K : \mathrm{val}(x) > 0\} \triangleleft \mathcal{O}_K$ unique maximal,
- $\pi \in K$ such that $\mathrm{val}(\pi) = 1$ is a uniformizer and $\mathfrak{m}_K = \mathcal{O}_K \pi$.

Let K be a field with a surjective valuation map

$$\mathrm{val} : K \to \mathbb{Z} \cup \{\infty\}.$$

Denote

- $\mathcal{O}_K = \{x \in K : \mathrm{val}(x) \geq 0\}$ is the valuation ring of K,
- $\mathfrak{m}_K = \{x \in K : \mathrm{val}(x) > 0\} \triangleleft \mathcal{O}_K$ unique maximal,
- $\pi \in K$ such that $\mathrm{val}(\pi) = 1$ is a uniformizer and $\mathfrak{m}_K = \mathcal{O}_K \pi$.

The valuation val can be extended to K^d or $K^{d \times d}$ coordinate-wise:

$$\mathrm{val}_3(2, 15, -1/36) = (0, 1, -2), \text{ for } K = \mathbb{Q}$$

$$\mathrm{val}_t \begin{pmatrix} 0 & t^{-5} + t^{-1} \\ -1/3 & 87t^7 - t^{11} \end{pmatrix} = \begin{pmatrix} \infty & -5 \\ 0 & 7 \end{pmatrix}, \text{ for } K = \mathbb{Q}((t))$$

Definition.

- A ($\mathcal{O}_K$-)lattice in K^d is a free $\mathcal{O}_K$-submodule L of rank d.
- An order (in $K^{d\times d}$) is a lattice Λ that is also a ring.
- A Λ-lattice is a lattice L with $\Lambda L \subseteq L$, i.e. L is Λ-stable.

Definition.

- A ($\mathcal{O}_K$-)lattice in K^d is a free $\mathcal{O}_K$-submodule L of rank d.
- An order (in $K^{d\times d}$) is a lattice Λ that is also a ring.
- A Λ-lattice is a lattice L with $\Lambda L \subseteq L$, i.e. L is Λ-stable.

Example.

- $\mathcal{O}_K^d$ is the standard lattice in K^d,
- $\mathcal{O}_K G$ is an order in KG,
- the stable lattices of the order

$$\Lambda(J_2) = \begin{pmatrix} \mathcal{O}_K & \pi\mathcal{O}_K \\ \pi\mathcal{O}_K & \mathcal{O}_K \end{pmatrix} \subseteq K^{2\times 2}$$

are $L_{(1,0)+\alpha(1,1)} = \mathcal{O}_K \pi^{1+\alpha} e_1 \oplus \mathcal{O}_K \pi^{\alpha} e_2$ for $\alpha \in \mathbb{Z}$ and ...

Definition.

- A $(\mathcal{O}_K$-)lattice in K^d is a free $\mathcal{O}_K$-submodule L of rank d.
- An order (in $K^{d\times d}$) is a lattice Λ that is also a ring.
- A Λ-lattice is a lattice L with $\Lambda L \subseteq L$, i.e. L is Λ-stable.

Example.

- $\mathcal{O}_K^d$ is the standard lattice in K^d,
- if G is a finite group, then $\mathcal{O}_K G$ is an order in KG,
- the stable lattices of the order

$$\Lambda(J_2) = \begin{pmatrix} \mathcal{O}_K & \pi\mathcal{O}_K \\ \pi\mathcal{O}_K & \mathcal{O}_K \end{pmatrix} \subseteq K^{2\times 2}$$

are $L_{(1,0)+\alpha(1,1)}$ and $L_{\beta(1,1)}$ and $L_{(0,1)+\gamma(1,1)}$ for $\alpha, \beta, \gamma \in \mathbb{Z}$.

Let Λ be an order in $K^{d\times d}$ and let L be a Λ-lattice. Then

- for each $n \in \mathbb{Z}$, also $\pi^n L$ is Λ-stable.

$$
\begin{aligned}
L_{(1,0)+\alpha(1,1)} &= \mathcal{O}_K \pi^{1+\alpha} e_1 \oplus \mathcal{O}_K \pi^{\alpha} e_2 = \pi^{\alpha} L_{(1,0)}, \\
L_{(0,0)+\beta(1,1)} &= \mathcal{O}_K \pi^{\beta} e_1 \oplus \mathcal{O}_K \pi^{\beta} e_2 = \pi^{\beta} L_{(0,0)}, \\
L_{(0,1)+\gamma(1,1)} &= \mathcal{O}_K \pi^{\gamma} e_1 \oplus \mathcal{O}_K \pi^{1+\gamma} e_2 = \pi^{\gamma} L_{(0,1)}.
\end{aligned}
$$

- if L' is Λ-stable, then so are $L \cap L'$ and $L + L'$.

Definition.

- Lattices with $L' = \pi^n L$ are called homothetic, denoted $L \sim L'$
- Lattices of the form L_u are called diagonal and all have compatible bases

Let $M = (m_{ij}) \in \mathbb{Z}^{d\times d}$. Then the set

$$\Lambda_M = \{X \in K^{d\times d} : \mathrm{val}(X) \geq M\}$$

is an $\mathcal{O}_K$-module, thanks to the defining properties of val.

The study of graduated orders was pioneered by Plesken and Zassenhaus (1983).

Let $M = (m_{ij}) \in \mathbb{Z}^{d\times d}$. Then the set

$$\Lambda_M = \{X \in K^{d\times d} : \mathrm{val}(X) \geq M\}$$

is an $\mathcal{O}_K$-module, thanks to the defining properties of val.

The study of graduated orders was pioneered by Plesken and Zassenhaus (1983).

Remark.

- Λ_M has maximal rank d^2 as a free $\mathcal{O}_K$-submodule of $K^{d\times d}$, because the entries of M are integers.
- Λ_M lives in a ring.

Question. When is Λ_M multiplicatively closed?

Example. For $K = \mathbb{Q}$, $d = 3$, and $\mathrm{val} = \mathrm{val}_p$:

$$\mathrm{val}_p \underbrace{\begin{pmatrix} 1 & 1 & p \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{M} \text{ but } \begin{pmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} = \mathrm{val} \underbrace{\begin{pmatrix} 2+p & 2+p & 1+2p \\ 3 & 3 & 2+p \\ 3 & 3 & 2+p \end{pmatrix}}_{X^2}$$

so Λ_M is not a ring.

Example. For $K = \mathbb{Q}$, $d = 3$, and $\mathrm{val} = \mathrm{val}_p$:

$$\mathrm{val}_p \underbrace{\begin{pmatrix} 1 & 1 & p \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{M} \text{ but } \begin{pmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} = \mathrm{val} \underbrace{\begin{pmatrix} 2+p & 2+p & 1+2p \\ 3 & 3 & 2+p \\ 3 & 3 & 2+p \end{pmatrix}}_{X^2}$$

so Λ_M is not a ring.

Example. $M = 0 \Rightarrow \Lambda_M = \mathcal{O}_K^{d\times d}$ is a maximal order in $K^{d\times d}$.

Remark. Orders of the form Λ_M are called graduated, tiled, split or monomial by different authors.

Example. For $K = \mathbb{Q}$, $d = 3$, and $\mathrm{val} = \mathrm{val}_p$:

$$\mathrm{val}_p \underbrace{\begin{pmatrix} 1 & 1 & p \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{M} \text{ but } \begin{pmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix} = \mathrm{val} \underbrace{\begin{pmatrix} 2{+}p & 2{+}p & 1{+}2p \\ 3 & 3 & 2{+}p \\ 3 & 3 & 2{+}p \end{pmatrix}}_{X^2}$$

so Λ_M is not a ring.

Example. $M = 0 \Rightarrow \Lambda_M = \mathcal{O}_K^{d\times d}$ is a maximal order in $K^{d\times d}$.

Remark. Orders of the form Λ_M are called graduated, tiled, split or monomial by different authors.

Proposition (Plesken). Λ_M is an order if and only if

$$m_{ii} = 0,\ m_{ij} + m_{jk} \geq m_{ik} \text{ for } 1 \leq i, j, k \leq d.$$

Theorem (Plesken). A lattice L in K^d is stable under Λ_M if and only if there exists $u \in \mathbb{Z}^d$ with

$$u_i - u_j \leq m_{ij} \quad \text{for} \quad 1 \leq i, j \leq d,$$

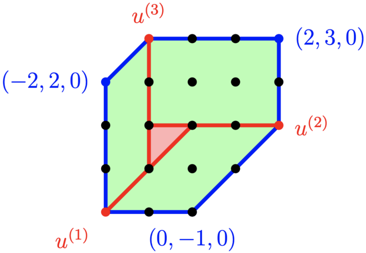
such that $L = L_u = \mathcal{O}_K \pi^{u_1} e_1 \oplus \ldots \oplus \mathcal{O}_K \pi^{u_d} e_d$. Moreover, two stable lattices L_u and $L_{u'}$ are isomorphic as Λ_M-modules if and only if there exists $n \in \mathbb{Z}$ such that $L_{u'} = \pi^n L_u$.

Theorem (Plesken). A lattice L in K^d is stable under Λ_M if and only if there exists $u \in \mathbb{Z}^d$ with

$$u_i - u_j \leq m_{ij} \quad \text{for} \quad 1 \leq i, j \leq d,$$

such that $L = L_u = \mathcal{O}_K \pi^{u_1} e_1 \oplus \ldots \oplus \mathcal{O}_K \pi^{u_d} e_d$. Moreover, two stable lattices L_u and $L_{u'}$ are isomorphic as Λ_M-modules if and only if there exists $n \in \mathbb{Z}$ such that $L_{u'} = \pi^n L_u$.

Example.

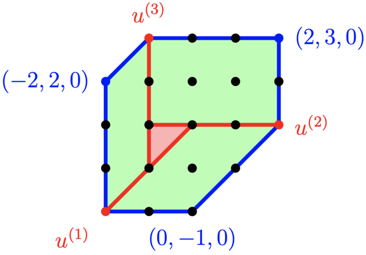


Here

$$M = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}$$

and dots represent eq. classes of Λ_M-lattices.

Example.



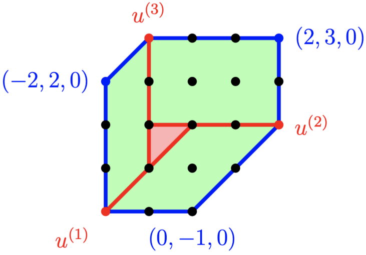
Here

$$M = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}$$

and dots represent eq. classes of Λ_M-lattices.

Definition. $Q_M = \{[u] \in \mathbb{R}^d/\mathbb{R}\mathbf{1} : u_i - u_j \leq m_{ij}\}$ is a polytrope.

Example.



Here

$$M = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}$$

and dots represent eq. classes of Λ_M-lattices.

Definition. $Q_M = \{[u] \in \mathbb{R}^d/\mathbb{R}\mathbf{1} : u_i - u_j \leq m_{ij}\}$ is a polytrope.

Theorem (Plesken). The following is a well-defined bijection:

$$Q_M \cap (\mathbb{Z}^d/\mathbb{Z}\mathbf{1}) \longrightarrow \{[L] : L \text{ is } \Lambda_M\text{-stable}\}$$
$$[u] \longmapsto [L_u]$$

With the following operations on $\mathbb{R}$ (which extend to $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$)

$$a \,\underline{\oplus}\, b = \min\{a, b\}, \quad a \,\overline{\oplus}\, b = \max\{a, b\}, \quad a \odot b = a + b$$

we have the following tropical picture:

- Λ_M is an order if and only if $M \,\underline{\odot}\, M = M$
- L_u is a stable lattice if and only if $M \,\underline{\odot}\, u^{\mathrm{t}} \geq u^{\mathrm{t}}$
- $L_u \cap L_{u'} = L_{u \,\overline{\oplus}\, u'}$ and $L_u + L_{u'} = L_{u \,\underline{\oplus}\, u'}$

With the following operations on $\mathbb{R}$ (which extend to $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$)

$$a \underline{\oplus} b = \min\{a,b\}, \quad a \overline{\oplus} b = \max\{a,b\}, \quad a \odot b = a+b$$

we have the following tropical picture:

- Λ_M is an order if and only if $M \underline{\odot} M = M$
- L_u is a stable lattice if and only if $M \underline{\odot} u^{\mathrm{t}} \geq u^{\mathrm{t}}$
- $L_u \cap L_{u'} = L_{u \overline{\oplus} u'}$ and $L_u + L_{u'} = L_{u \underline{\oplus} u'}$

Definition.

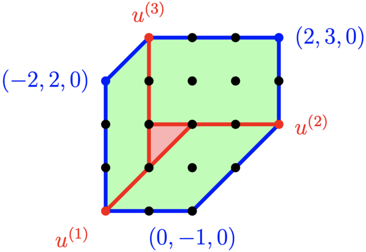
- $\mathcal{P}_d = \{N \in \mathbb{R}_0^{d\times d} : N \underline{\odot} N = N\}$ is the polytrope region
- $\mathcal{P}_d(M) = \{N \in \mathcal{P}_d : N \leq M\}$ is the truncated poly region

Then $\mathcal{P}_d$ is a (d^2-d)-dimensional convex polyhedral cone and $\mathcal{P}_d(M)$ parametrizes subpolytropes of Q_M eq. overorders of Λ_M.

Definition. $M \in \mathcal{P}_d$ is in standard form if $m_{ij} + m_{ji} > 0$ $(i \neq j)$.

Definition. $M \in \mathcal{P}_d$ is in standard form if $m_{ij} + m_{ji} > 0$ $(i \neq j)$.

Theorem. Let $M \in \mathcal{P}_d$ be in standard form. Then Q_M is both a min-plus and a max-plus simplex. The min-plus vertices u are the columns of M and represent L_u's that are projective Λ_M-modules. The max-plus vertices v are the columns of $-M^t$, and they represent the injective Λ_M-modules L_v.

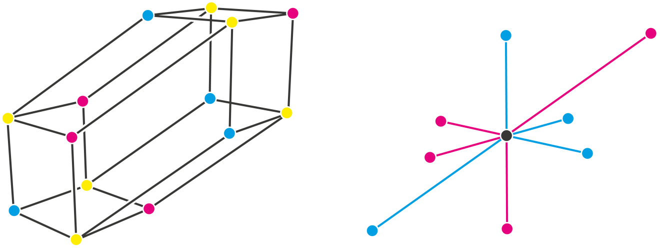


Recall that

$$M = \begin{pmatrix} 0 & 1 & 2 \\ 4 & 0 & 3 \\ 2 & 1 & 0 \end{pmatrix}.$$

Definition. $M \in \mathcal{P}_d$ is in standard form if $m_{ij} + m_{ji} > 0$ $(i \neq j)$.

Theorem. Let $M \in \mathcal{P}_d$ be in standard form. Then Q_M is both a min-plus and a max-plus simplex. The min-plus vertices u are the columns of M and represent L_u's that are projective Λ_M-modules. The max-plus vertices v are the columns of $-M^t$, and they represent the injective Λ_M-modules L_v.

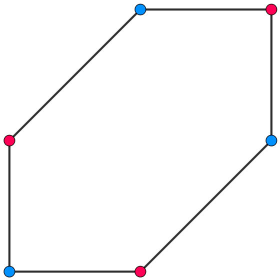


Here $d = 4$ and $M = J_4$ has all 1's outside the diagonal.

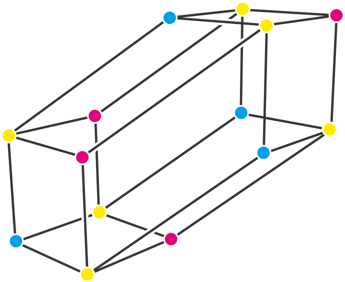
The tropical distance on $\mathbb{R}^d/\mathbb{R}\mathbf{1}$ is given by

$$\mathrm{dist}(u,v) = \max_{1\le i\le d}(u_i - v_i) - \min_{1\le j\le d}(u_j - v_j).$$

For each $r \geq 0$, we have that $\mathrm{B}_r(0) = Q_{rJ_d}$, where $J_d \in \mathbb{Z}^{d\times d}$ has 0's on the main diagonal and all 1's outside it.



$\mathrm{B}_1(0) = Q_{J_3}$



$\mathrm{B}_1(0) = Q_{J_4}$

Definition. The Bruhat-Tits building $\mathcal{B}_d(K)$ is a simplicial complex where:

- the vertices are equivalence classes of lattices in K^d,
- $([L_1], \ldots, [L_s])$ is a simplex if $L_1 \supset L_2 \supset \ldots L_s \supset \pi L_1$ (up to reordering and picking representatives)

Definition. The Bruhat-Tits building $\mathcal{B}_d(K)$ is a simplicial complex where:

- the vertices are equivalence classes of lattices in K^d,
- $([L_1], \ldots, [L_s])$ is a simplex if $L_1 \supset L_2 \supset \ldots L_s \supset \pi L_1$ (up to reordering and picking representatives)

Remark. From the point of view of the building, looking at diagonal lattices (eq. graduated orders) is the same as working in one apartment $\mathcal{A}$ (compatible bases)

Remark. If Λ is an order, then $Q(\Lambda) = \{\text{stable } \Lambda\text{-lattices}\}$ is non-empty, convex, and bounded in $\mathcal{B}_d(K)$.

Definition. The Bruhat-Tits building $\mathcal{B}_d(K)$ is a simplicial complex where:

- the vertices are equivalence classes of lattices in K^d,
- $([L_1], \ldots, [L_s])$ is a simplex if $L_1 \supset L_2 \supset \ldots L_s \supset \pi L_1$ (up to reordering and picking representatives)

Remark. From the point of view of the building, looking at diagonal lattices (eq. graduated orders) is the same as working in one apartment $\mathcal{A}$ (compatible bases)

Remark. If Λ is an order, then $Q(\Lambda) = \{\text{stable } \Lambda\text{-lattices}\}$ is non-empty, convex, and bounded in $\mathcal{B}_d(K)$.

Question. What's life like when you are not quarantined in one apartment?

An example for $d = 2$ and $d = 3$:

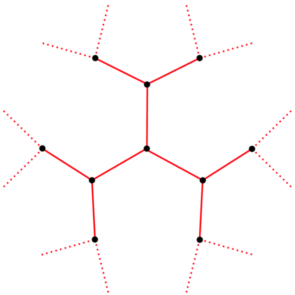
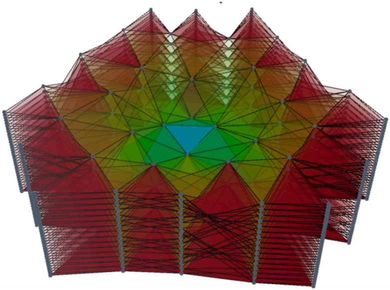


Figure 5. The building $\mathcal{B}(SL_2(\mathbb{Q}), \nu_2)$ is an infinite tree.



(b) The affine building $\mathcal{B}(SL_3(\mathbb{Q}), \nu_2)$ up to distance 5 from the chamber in the centre.

Bekker, Solleveld - The Buildings Gallery: visualising buildings (2021)

More in the online gallery: https://buildings.gallery

For L_1, L_2 lattices in K^d, define

$$\begin{aligned}\mathrm{dist}([L_1],[L_2]) &= \min\{s \mid \exists L_1' \in [L_1], L_2' \in [L_2],\ \pi^s L_1' \subseteq L_2' \subseteq L_1'\} \\ &= \min\{s \mid \exists m \text{ with } \pi^s L_1 \subseteq \pi^m L_2 \subseteq L_1\}\end{aligned}$$

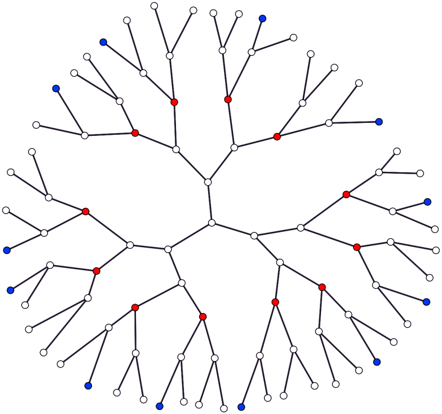
Then the following hold:

- dist agrees with the tropical distance in one apartment,
- if $d = 2$, then dist is the same as the graph distance.

For each $r \geq 0$, define

$$\mathrm{B}_r = \{[L] \mid \mathrm{dist}([\mathcal{O}_K^d],[L]) \leq r\}$$

and, choosing the appropriate basis, note that $\mathrm{B}_r \cap \mathcal{A} = Q_{rJ_d}$. A vertex of B_r is an element of $\{P \text{ vertex of } \mathrm{B}_r \cap \mathcal{A} \mid \mathcal{A} \text{ apartment}\}$.



This is B_5 inside of $\mathcal{B}_2(\mathbb{Q}_2)$. Every boundary point is a vertex.

A bolytrope with center $Q_M \subseteq \mathcal{A}$ and radius $r \geq 0$ is

$$\mathrm{B}_r(M) = \{[L] \mid \mathrm{dist}([L], Q_M) \leq r\}.$$

Then $\mathrm{B}_r(M) \cap \mathcal{A} = Q_{M+rJ_d}$

A bolytrope with center $Q_M \subseteq \mathcal{A}$ and radius $r \geq 0$ is

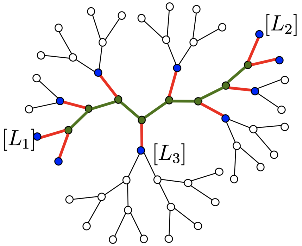
$$\mathrm{B}_r(M) = \{[L] \mid \mathrm{dist}([L], Q_M) \leq r\}.$$

Then $\mathrm{B}_r(M) \cap \mathcal{A} = Q_{M+rJ_d}$ and we define a bolytrope order to be an order of the form

$$\Lambda_r(M) = \{X \in \Lambda_{M+rJ_d} \mid X_{11} \equiv \ldots \equiv X_{dd} \bmod \pi^r\}.$$

Theorem. $Q(\Lambda_r(M)) = \mathrm{B}_r(M)$.

Theorem. If $d = 2$, then all closed orders are bolytrope orders.



Let $r > 0$ be an integer and define $\partial\,\mathrm{B}_r = \mathrm{B}_r \setminus \mathrm{B}_{r-1}$.

A spherical code in $\mathcal{B}_d(K)$ is a subset $\mathcal{C} \subseteq \partial\,\mathrm{B}_r$ with $|\mathcal{C}| \geq 2$.

Let $r > 0$ be an integer and define $\partial\,\mathrm{B}_r = \mathrm{B}_r \setminus \mathrm{B}_{r-1}$.

A spherical code in $\mathcal{B}_d(K)$ is a subset $\mathcal{C} \subseteq \partial\,\mathrm{B}_r$ with $|\mathcal{C}| \geq 2$.

Spherical codes in the Euclidean setting can be defined from sphere packings and have various applications in the field of telecommunication.

In view of these applications, it is desirable to produce sizeable codes of large internal distance and small length. Optimal codes have the "best possible" coexistence constraints on these parameters.

Let $r > 0$ be an integer and define $\partial\,\mathrm{B}_r = \mathrm{B}_r \setminus \mathrm{B}_{r-1}$.

A spherical code in $\mathcal{B}_d(K)$ is a subset $\mathcal{C} \subseteq \partial\,\mathrm{B}_r$ with $|\mathcal{C}| \geq 2$.

Spherical codes in the Euclidean setting can be defined from sphere packings and have various applications in the field of telecommunication.

In view of these applications, it is desirable to produce sizeable codes of large internal distance and small length. Optimal codes have the "best possible" coexistence constraints on these parameters.

The minimum distance of $\mathcal{C}$ is

$$\mathrm{dist}(\mathcal{C}) = \min\{\mathrm{dist}([L_1],[L_2]) \mid [L_1],[L_2] \in \mathcal{C}, [L_1] \neq [L_2]\}.$$

Define $V_r = \mathcal{O}_K^d / \pi^r \mathcal{O}_K^d$ – free $\mathcal{O}_K / \pi^r \mathcal{O}_K$-module of rank d.

Then the following hold:

- V_r is a vector space if and only if $r = 1$,
- identify $[L] \in \partial \mathrm{B}_r$ with $L \leq V_r$ with $\pi^{r-1} V_r \not\subseteq L \not\subset \pi V_r$,
- vertices of B_r represent free $\mathcal{O}_K / \pi^r \mathcal{O}_K$-submodules of V_r.

Define $V_r = \mathcal{O}_K^d / \pi^r \mathcal{O}_K^d$ – free $\mathcal{O}_K/\pi^r\mathcal{O}_K$-module of rank d.

Then the following hold:

- V_r is a vector space if and only if $r = 1$,
- identify $[L] \in \partial\,\mathrm{B}_r$ with $L \leq V_r$ with $\pi^{r-1}V_r \nsubseteq L \not\subset \pi V_r$,
- vertices of B_r represent free $\mathcal{O}_K/\pi^r\mathcal{O}_K$-submodules of V_r.

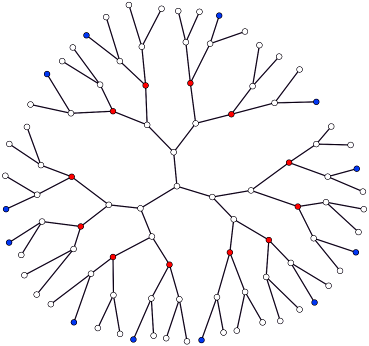
In particular, spherical codes in Bruhat-Tits buildings are instances of submodule codes over chain rings.

Definition. $\mathrm{Gr}(n, V_r) = \{$free $\mathcal{O}_K/\pi^r\mathcal{O}_K$-subs of V_r of rank $n\}$

Remark. If $r = 1$, then we recover the usual Grassmannian.

Definition. Let $1 \leq \alpha \leq r$. A Sperner code with parameters (d, r, α) is $\mathcal{C} \subseteq \mathrm{Gr}(\lceil d/2 \rceil, V_r)$ such that the following is a bijection:

$$\mathcal{C} \longrightarrow \mathrm{Gr}(\lceil d/2 \rceil, \pi^{\alpha-1} V_r), \quad L \longmapsto \pi^{\alpha-1} L.$$



The blue points form a Sperner code with parameters $(2, 5, 3)$.

Theorem. Let $1 \leq \alpha \leq r$ and let $\mathcal{C} \subseteq \mathrm{B}_r$ be a spherical code of maximal size with $\mathrm{dist}(\mathcal{C}) = 2\alpha$. Then

- $|\mathcal{C}| \geq |\,\mathrm{Gr}(\lceil d/2 \rceil, V_{r+1-\alpha})|$,
- if $d = 2$ or $\alpha = r$, then $|\mathcal{C}| = |\,\mathrm{Gr}(\lceil d/2 \rceil, V_{r+1-\alpha})|$.

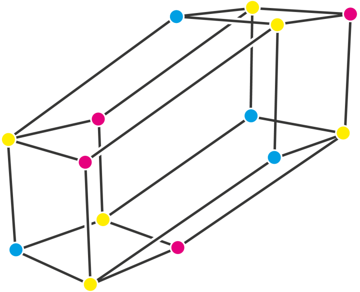
In particular, Sperner codes are optimal if $d = 2$ or $\alpha = r$.

Theorem. Let $1 \leq \alpha \leq r$ and let $\mathcal{C} \subseteq \mathrm{B}_r$ be a spherical code of maximal size with $\mathrm{dist}(\mathcal{C}) = 2\alpha$. Then

- $|\mathcal{C}| \geq |\,\mathrm{Gr}(\lceil d/2 \rceil, V_{r+1-\alpha})|$,
- if $d = 2$ or $\alpha = r$, then $|\mathcal{C}| = |\,\mathrm{Gr}(\lceil d/2 \rceil, V_{r+1-\alpha})|$.

In particular, Sperner codes are optimal if $d = 2$ or $\alpha = r$.

Remark. Codes in one apartment generalize permutation codes...



ArXiv identifiers: 2107.00503, 2111.11244, 2202.13370

And check out our Mathrepo page:
https://mathrepo.mis.mpg.de/OrdersPolytropes/index.html