(Strong) isomorphism of p-groups and orbit counting

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MAX-PLANCK-GESELLSCHAFT

CLASSIFYING FINITE GROUPS

Construction and Isomorpshim

In 1854, Cayley asked for a classification of finite groups of given order, up to isomorphism.

This question splits in two main parts: contruction and isomorphism testing.

To this, Magnus wrote in 1937 that...

"the main difficulty does not lie in the construction of groups of a given type, but rather in the specification of a complete set of representatives of the constructed groups"

Many authors involved: Cayley, Hall and Senior, Neubüser, Laue, Hölder, James, Newman, O'Brien, Betten, Besche, Eick, Vaughan-Lee, ...and many more!

Some starting ideas:

- multiplication tables,
- symmetries of objects,
- units of rings,
- operations on groups (e.g. $\times$, $\rtimes$, $\wr$, $\ldots$),
- extensions,...

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In particular, $E/N \cong G$ and, if N is abelian, G acts on N.

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Today's focus. Finite p-groups (iterated extensions by "just one" finite simple factor).

Example. Let p be a prime number. Then there are 5 groups of order p^3:

abelian | $\mathbb{Z}/(p)^3$, $\mathbb{Z}/(p) \times \mathbb{Z}/(p^2)$, $\mathbb{Z}/(p^3)$

nonabelian | $\begin{cases} \mathrm{D}_8,\ \mathrm{Q}_8 & \text{if } p = 2, \\ \mathrm{Heis}(\mathbb{F}_p),\ \mathbb{Z}/(p^2) \rtimes \mathbb{Z}/(p) & \text{otherwise,} \end{cases}$

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The nonabelian groups are extensions $1 \to \mathbb{F}_p \to E \to \mathbb{F}_p^2 \to 1$ and satisfy $[E, E] = \mathrm{Z}(E) \cong \mathbb{F}_p$. The p-groups satisfying $|[E, E]| = p$ have been classified by Blackburn (1999).

EXTENSIONS AND COHOMOLOGY

A recipe for classifying extensions up to . . .

Let N be a $\mathbb{Z}G$-module. Then

- $\mathrm{Z}^2(G;N)$ is the set of all maps $c : G \times G \to N$ such that

$$x \cdot c(y,z) + c(x,yz) = c(xy,z) + c(x,y).$$

- $\mathrm{B}^2(G;N) \subseteq \mathrm{Z}^2(G;N)$ consists of the elements with

$$\exists f : G \to N \text{ such that } c(x,y) = f(xy) - f(x) - x \cdot f(y).$$

- $\mathrm{H}^2(G;N) = \mathrm{Z}^2(G;N)/\mathrm{B}^2(G;N)$ – second cohomology group.

Let N be a trivial $\mathbb{Z}G$-module. Then

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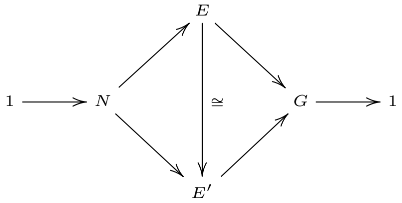
Elements: $[c] \in \mathrm{H}^2(G;N)$.

From now on we work with trivial actions!

The second cohomology group parametrizes central extensions up to equivalence:



where $E_c = G \times N$ and $(g,n)(g',n') = (gg', n + n' + c(g,g'))$.

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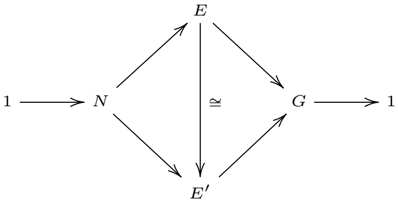

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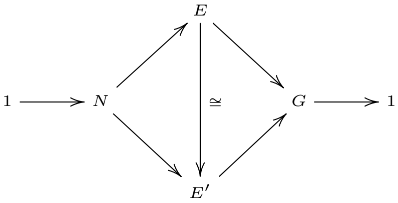

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Example.

- $E = G \times N \iff [c] = 0$ (measure of obstruction)
- $\mathbb{Z}/(p)$ has only two extensions by itself: $\mathbb{Z}/(p^2)$ and $\mathbb{Z}/(p)^2$ but $\mathrm{H}^2(\mathbb{Z}/(p); \mathbb{Z}/(p))$ has p elements!

Theorem (Besche-Eick, 1999)

Let N be a $\mathbb{Z}G$-module. Then the strong isomorphism classes of extensions of G by N are naturally parametrized by the action of the compatible pairs of $A = \mathrm{Aut}(G) \times \mathrm{Aut}(N)$ on $\mathrm{H}^2(G; N)$:

$$(\sigma, \lambda)c : (x, y) \mapsto \lambda c(\sigma^{-1}(x), \sigma^{-1}(y)).$$

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Strong isomorphism (Fitting, 1936): (include equivalence)

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Compatible pairs: introduced by Robinson (1981) for fixed modules, computationally extremely useful. (For non-fixed modules: Laue)

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Note: $\#\{$iso classes$\} \leq \#\{$s-iso classes$\} \leq \#\{$eq classes$\}$.

Corollary

Let N be a trivial $\mathbb{F}_p G$-module. Then: $((\sigma, \lambda)[c] = [\lambda c \sigma^{-1}])$

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Recall. If $G=\mathbb{Z}/(p)$, we had two extensions of G by $\mathbb{F}_p$, corresponding to $[0]$ and $\{\lambda[c]:\lambda\in\mathbb{F}_p^*\}$.

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In general. Non-trivial A-orbits in $\mathrm{H}^2(G; \mathbb{F}_p)$ have order divisible by $p - 1 \rightsquigarrow$ one can "work projectively".

Assumptions:

- p prime number
- G finite abelian p-groups, with no summands of order 2
- $\exp(G) = p^n$
- $V = G/pG$ and $\dim(V) = \mathrm{d}(G) = r + 1$
- $A = \mathrm{Aut}(G) \times \mathbb{Z}_p^*$ – usually $A = \mathrm{Aut}(G) \times \mathbb{F}_p^*$ will do

Our goal today: determining the orbits of the action of A on $\mathrm{H}^2(G; \mathbb{F}_p)$.

Our hope for the future: develop techniques that will generalize to modules N that are "hard to deal with" – for example cyclic groups, . . .

Under our assumptions, $\mathrm{H}^2(G;\mathbb{F}_p)$ is an $\mathbb{F}_p$-vector space with a (canonical!) split short exact sequence

$$0 \to \mathrm{Ext}^1(G,\mathbb{F}_p) \to \mathrm{H}^2(G;\mathbb{F}_p) \to \mathrm{Hom}(\Lambda^2 V,\mathbb{F}_p) \to 0$$

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Another description: $\mathrm{H}^2(G;\mathbb{F}_p) = \mathrm{H}^2_{\mathrm{ab}}(G;\mathbb{F}_p) \oplus \langle \mathrm{Im}\cup\rangle$, where

$$\cup : \mathrm{Hom}(G,\mathbb{F}_p) \times \mathrm{Hom}(G;\mathbb{F}_p) \to \mathrm{H}^2(G;\mathbb{F}_p)$$

and $f \cup g : G \times G \to \mathbb{F}_p$ is defined by $(x,y) \mapsto f(x)g(y)$.

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Observation.

- G cyclic $\Rightarrow \mathrm{H}^2(G;\mathbb{F}_p) = \mathrm{H}^2_{\mathrm{ab}}(G;\mathbb{F}_p)$
- $\mathrm{Im} \cup = \langle \mathrm{Im} \cup \rangle \iff \mathrm{d}(G) \leq 3$.

COHOMOLOGY & GEOMETRY

A very useful correspondence

Define

- $\mathcal{S}^i(G) = \{N \leq G \mid G/N \cong \mathbb{F}_p^i\}$,
- $\mathcal{G}(i,V) = \{W \leq V \mid \dim(V/W) = i\}$.

Proposition

There are isomorphisms of A-sets ($\mathrm{Aut}(G)$-sets)

$$\mathbb{P}\,\mathrm{H}^2_{\mathrm{ab}}(G;\mathbb{F}_p) \times \mathbb{P}\,\mathrm{Im}\cup \longleftrightarrow \mathcal{S}^1(G) \times \mathcal{S}^2(G) \longleftrightarrow \mathcal{G}(1,V) \times \mathcal{G}(2,V)$$

What do I mean by this?

$$\begin{array}{ccc}
([c],[\omega]) & \longmapsto & (\sigma,\lambda)([c],[\omega]) = ([c\sigma^{-1},\omega\sigma^{-1}]) \\
\updownarrow & & \updownarrow \\
(T,M) & \longmapsto & (\sigma,\lambda)(T,M) = (\sigma(T),\sigma(M)) \\
\updownarrow & & \updownarrow \\
(\overline{T},\overline{M}) & \longmapsto & (\sigma,\lambda)(\overline{T},\overline{M}) = (\overline{\sigma}(\overline{T}),\overline{\sigma}(\overline{M}))
\end{array}$$

IDENTIFY!

Grassmannians ⟩ An example

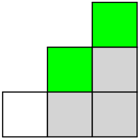
Let $G = \mathbb{Z}/(p) \oplus \mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p^3)$ and

$$\phi : \mathbb{Z}/(p) \to \mathbb{Z}/(p^2), \quad X \bmod p \mapsto X \bmod p^2.$$

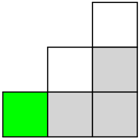
We define $[c] \in \mathrm{H}^2_{\mathrm{ab}}(G; \mathbb{F}_p)$ and $[\omega] \in \mathrm{Im} \cup$ by

$$\begin{aligned} c : G \times G \to \mathbb{F}_p, &\quad (x, y) \mapsto \phi(x_1 + y_1) - \phi(x_1) - \phi(y_1) \\ \omega : G \times G \to \mathbb{F}_p, &\quad (x, y) \mapsto x_2 y_3 \bmod p. \end{aligned}$$

Then we have



$[c] \longleftrightarrow T = \ker \pi_1 + pG$



$[\omega] \longleftrightarrow M = (\ker \pi_2 \cap \ker \pi_3) + pG$

SUBGROUP LEVELS

The relative inclusion of pairs of subgroups

Let M and T be subgroups of G.

Definition

The T-levels of M are $\ell\mathrm{L}_T(M) = (\ell_T(M), \mathrm{L}_T(M))$ where

- $\ell_T(M) = 1 + \max\{0 \leq i \leq \log_p \exp(T) \,:\, T[p^i] \subseteq M \cap T\}$,
- $\mathrm{L}_T(M) = \min\{j \in \mathbb{Z}_{\geq 0} \,:\, T[p^j] + (M \cap T) = T\}$.

If $T = G$, simply write $\ell\mathrm{L}(M) = (\ell(M), \mathrm{L}(M))$ for $\ell\mathrm{L}_G(M)$.

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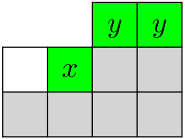
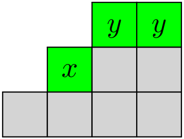
Example.

- $\ell\mathrm{L}(G) = (n + 1, 0)$
- M maximal in $G \Rightarrow \ell(M) = \mathrm{L}(M)$

Here $G = \langle g_1, g_2, g_3, g_4 \rangle$, $pG = \textcolor{lightgray}{\bullet}$, $T = \textcolor{green}{\bullet} + \textcolor{lightgray}{\bullet}$, $M = \langle x, y \rangle + \textcolor{lightgray}{\bullet}$.

Example. $G = [2, 2, 3, 3]$, $x = g_2$, $y = g_4 - g_3$.





- $\ell\mathrm{L}(M) = (2, 3)$
- $\ell\mathrm{L}_T(M) = (3, 3)$
- $i_T(M) = 0$

Subgroup Levels 〉 Another example

Here $G = \langle g_1, g_2, g_3, g_4 \rangle$, $pG = \bullet$, $T = \bullet + \bullet$, $M = \langle x, y \rangle + \bullet$.

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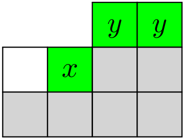

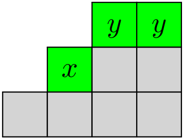

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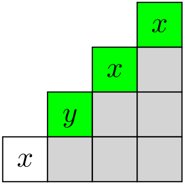
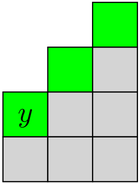
Example. $G = [1, 2, 3, 4]$, $x = g_4 - g_3 - g_1$, $y = g_2$.





- $\ell\mathrm{L}(M) = (1, 3)$
- $\ell\mathrm{L}_T(M) = (3, 4)$
- $i_T(M) = 1$

OUR MAIN RESULT

Cohomology, orbits, and levels

Theorem

Let $[c], [d] \in \mathrm{H}^2_{\mathrm{ab}}(G; \mathbb{F}_p)$ *and* $[\omega], [\vartheta] \in \mathrm{Im}\cup$. *Then*

$$[c] + [\omega] \sim_A [d] + [\vartheta]$$

$$\Updownarrow$$

$$(\ell\mathrm{L}[c], \ell\mathrm{L}[\omega], \ell\mathrm{L}_c[\omega], i_c[\omega]) = (\ell\mathrm{L}[d], \ell\mathrm{L}[\vartheta], \ell\mathrm{L}_d[\vartheta], i_d[\vartheta]).$$

Structure of the proof.

- $[\omega] = [\vartheta] = 0 \rightsquigarrow$ Abelian extensions.
- $[c] = [d] = 0 \rightsquigarrow$ A-orbits in $\mathrm{Im}\cup$.
- General $\rightsquigarrow$ $A_{c/d}$-orbits in $\mathrm{Im}\cup$: $i = 0$ vs. $i = 1$.

Write $G = \mathbb{Z}/(p^{m_1}) \oplus \mathbb{Z}/(p^{m_2}) \oplus \mathbb{Z}/(p^{m_3})$ with $m_1 \leq m_2 \leq m_3$.
Then, up to isomorphism, there are

$$\begin{cases} 1 & \text{if } 0 = m_1 = m_2 = m_3, \\ 2 & \text{if } 0 = m_1 = m_2 < m_3, \\ 4 & \text{if } 0 = m_1 < m_2 = m_3, \\ 6 & \text{if } 0 = m_1 < m_2 < m_3, \\ 5 & \text{if } 0 < m_1 = m_2 = m_3, \\ 11 & \text{if } 0 < m_1 < m_2 = m_3, \\ 11 & \text{if } 0 < m_1 = m_2 < m_3, \\ 19 & \text{if } 0 < m_1 < m_2 < m_3, \end{cases}$$

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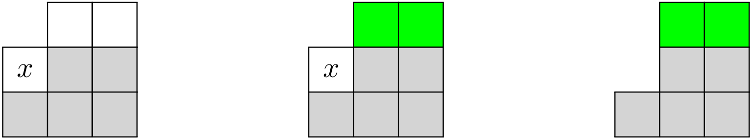
Assume $0 < m_1 < m_2 = m_3$. This is the table of levels:

	$[\omega] = 0$	$[\omega] \neq 0$
$[c] = 0$	$(m_2+1, 0 \mid m_2+1, 0 \mid m_2+1, 0 \mid 0)$	$(m_2+1, 0 \mid m_1, m_2 \mid m_1, m_2 \mid 0)$
		$(m_2+1, 0 \mid m_2, m_2 \mid m_2, m_2 \mid 0)$
$[c_1] = [\beta(\gamma_1^*)]$	$(m_1, m_1 \mid m_2+1, 0 \mid m_2+1, 0 \mid 1)$	$(m_1, m_1 \mid m_1, m_2 \mid m_2, m_2 \mid 0)$
		$(m_1, m_1 \mid m_1, m_2 \mid m_2, m_2 \mid 1)$
		$(m_1, m_1 \mid m_2, m_2 \mid m_2, m_2 \mid 1)$
$[c_2] = [\beta(\gamma_2^*)]$	$(m_2, m_2 \mid m_2+1, 0 \mid m_2+1, 0 \mid 1)$	$(m_2, m_2 \mid m_1, m_2 \mid m_1, m_1 \mid 0)$
		$(m_2, m_2 \mid m_1, m_2 \mid m_1, m_2 \mid 1)$
		$(m_2, m_2 \mid m_2, m_2 \mid m_2, m_2 \mid 0)$

With associated orbit sizes:

$$(1, p-1, p^3-p \mid p^3-p, p-1 \mid (p-1)(p^2-1), (p-1)^2), (p-1)(p^3-p^2-p+1) \mid$$
$$(p^3-p)(p^2-p), (p^3-p)(p-1), (p^3-p)(p^3-p^2))$$

Missing. For example $(m_1, m_1 \mid m_2, m_2 \mid m_2, m_2 \mid 0)$



- Complete determination of orbit sizes when $\mathrm{d}(G) \leq 3$.
- Complete "level vectors" needed as soon as $\mathrm{d}(G) \geq 4$.
- In general, lower bounds for the number of orbits and orbit sizes for extensions with "big centre".
- It can be shown that "$\mathrm{Z}(E)/[E,E] = M$". Can T be interpreted in a similar way?
- The orbit sizes are polynomial in p. Always?
- Generalization of levels to describe subgroups of G (containing pG)? Classical?
- Description of elements of $\mathbb{P}\,\mathrm{H}^2(G;\mathbb{F}_p)$ in terms of the geometry of $\mathbb{P}(\Lambda^2 V)$? (Secant varieties, up to...?)
- What happens beyond $\mathbb{F}_p$?


thank you

The SES $0 \to \mathbb{F}_p \to \mathbb{Z}/(p^{n+1}) \to \mathbb{Z}/(p^n) \to 0$ yields a long exact sequence of A-groups

$$\ldots \to \mathrm{Hom}(G, \mathbb{Z}/(p^{n+1})) \to \mathrm{Hom}(G, \mathbb{Z}/(p^n)) \xrightarrow{\beta} \mathrm{H}^2(G; \mathbb{F}_p) \to \ldots$$

One shows:

- $\mathrm{Im}\,\beta = \mathrm{H}^2_{\mathrm{ab}}(G; \mathbb{F}_p)$
- to study the A-orbits of $\mathrm{H}^2_{\mathrm{ab}}(G; \mathbb{F}_p)$, we can look at A-orbits of $\mathrm{Hom}(G, \mathbb{Z}/(p^n))/\ker\beta$ – Bockquivalence classes

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Write

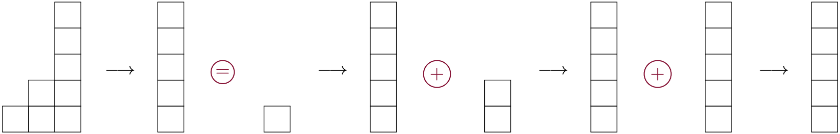
$$G = \bigoplus_{j=1}^{t} \bigoplus_{k=1}^{r_t} \langle \gamma_{jk} \rangle \text{ with } |\gamma_{jk}| = |\gamma_{jh}| < |\gamma_{(j+1)l}|.$$

Proofs 〉 Abelian extensions

Then we have

$$\widehat{G} = \mathrm{Hom}(G, \mathbb{Z}/(p^n)) = \bigoplus_{j=1}^{t} \bigoplus_{k=1}^{r_t} \langle \gamma_{jk}^* \rangle \text{ with } \gamma_{j,k}^*(\gamma_{ih}) = \delta_{(j,k),(i,h)}$$

Example.



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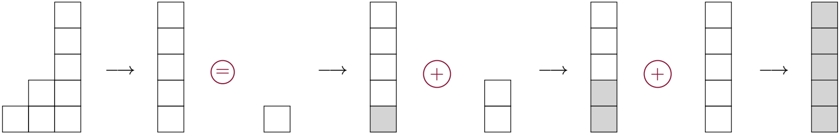

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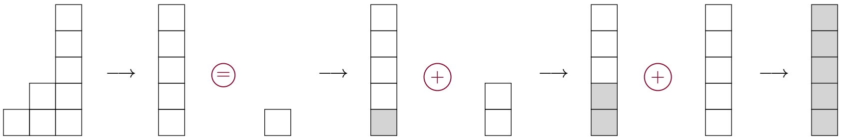


Observation.

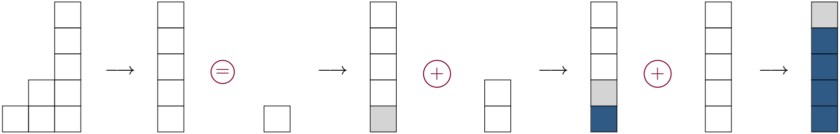
$$A \cdot [0] = \ker \beta = \{\sum_{j=1}^{t} \sum_{k=1}^{r_j} \alpha_{jk} \gamma_{jk}^* \mid \alpha_{jk} \in p\mathbb{Z}_p\}$$

Proofs 〉 Abelian extensions

Recall

$$\ldots \to \mathrm{Hom}(G, \mathbb{Z}/(p^{n+1})) \to \mathrm{Hom}(G, \mathbb{Z}/(p^n)) \xrightarrow{\beta} \mathrm{H}^2(G; \mathbb{F}_p) \to \ldots$$

Example.



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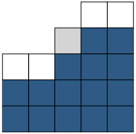
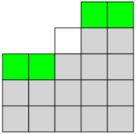
Proposition

If $c \in \widehat{G}$ satisfies $|\operatorname{Im} \widehat{\pi}_j(c)| = p^{n_j}$ and, for each $l > j$, one has $|\operatorname{Im} \widehat{\pi}_j(c)| < p^{n_l}$, then $c \approx_A \gamma_{j1}^*$.



Here

- $j = 2$
- $\ell\mathrm{L}(c) = \ell\mathrm{L}(T) = (4, 4)$



Moreover, there are $t + 1$ (here, $= 3 + 1 = 4$) Bockquivalence classes.

Proofs 〉 The cup product

Identify $[c]+[\omega]=(T_c,M_\omega)$ and $[d]+[\vartheta]=(T_d,M_\vartheta)$.

- If $[c]=[d]=0$, show that (scalars!)

$$[\omega]\sim_A[\vartheta]\iff M_\omega\sim_{\mathrm{Aut}(G)}M_\vartheta\iff \ell\mathrm{L}(M_\omega)=\ell\mathrm{L}(M_\vartheta).$$

- Use abelian classification to reduce to $[c]+[\omega]$ vs. $[c]+[\vartheta']$.
- Observe that $(\lambda,\lambda)[c]+[\omega]=[c]+\lambda[\omega]$.
- Show that

$$(T_c,M_\omega)\sim_{\mathrm{Aut}(G)}(T_c,M_{\vartheta'})$$

$$\Updownarrow$$

$$(\ell\mathrm{L}[c],\ell\mathrm{L}[\omega],\ell\mathrm{L}_c[\omega],i_c[\omega])=(\ell\mathrm{L}[d],\ell\mathrm{L}[\vartheta],\ell\mathrm{L}_d[\vartheta],i_d[\vartheta]).$$