EVOLVING GROUPS

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- $\gcd\{|J : I|, p\} = 1$
- $|G : J|$ is a p-power.

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Example:

- The only p-evolution of a Sylow subgroup is G.
- If $G = \mathrm{A}_5$ then the trivial subgroup has a 5-evolution in G, i.e. A_4, but it has no 2-evolution.

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Lemma

G evolving, $N \lhd G \Rightarrow N$ and G/N are evolving.

Cohomology

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Theorem

The following are equivalent.

- *For every G-module M, integer q, and $c \in \widehat{\mathrm{H}}^q(G, M)$, the minimum of the set $\{|G : H| \mid H \leq G \text{ with } c \in \ker \mathrm{Res}^G_H\}$ coincides with its greatest common divisor.*
- *The group G is an evolving group.*

A group G is *supersolvable* if there exists a chain $1 = N_0 \leq N_1 \leq \ldots \leq N_t = G$ such that

- $N_i \lhd G$
- N_{i+1}/N_i is cyclic.

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Theorem

Let G be an evolving group. Then G is supersolvable and it is isomorphic to the semidirect product of two nilpotent groups of coprime orders.

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Lemma

- *Every supersolvable group has a Sylow family.*
- *Sylow families are unique up to conjugation.*

Assume G has a Sylow family $(S_q)_{q\in\mathcal{P}}$ and let $T_p = \langle S_q \mid q < p \rangle$;

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Theorem

The following are equivalent.

- *The group G is an evolving group.*
- *The group G is a supersolvable group with the correction property.*

Let G be a group with a Sylow family $(S_p)_{p\in\mathcal{P}}$.

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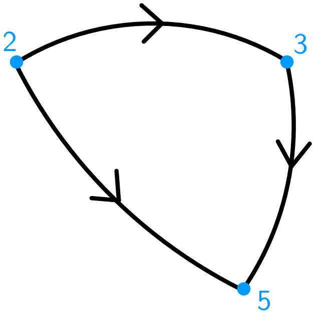
Lemma

If G is an evolving group and $p \in \mathcal{P}$, then T_p acts on $S_p/\Phi(S_p)$ by scalar multiplications.

Example: Graph associated to $G = \mathbb{F}_{25} \rtimes (\mu_3(\mathbb{F}_{25}) \rtimes \mathrm{Aut}(\mathbb{F}_{25}))$.



Call $\mathcal{S}$ the set of source vertices, $\mathcal{T}$ the set of target vertices and $\mathcal{I}$ the set of isolated vertices of the graph.

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Theorem

Let G be an evolving group and let $(S_p)_{p\in\mathcal{P}}$ be a Sylow family. Then $\mathcal{S}\cap\mathcal{T}=\varnothing$ and

$$G=\left(\prod_{p\in\mathcal{T}} S_p \rtimes \prod_{p\in\mathcal{S}} S_p\right)\times\prod_{p\in\mathcal{I}} S_p.$$

Let $p > 2$ be a prime number. Then the group

$$G = \left\{ \begin{pmatrix} h & u & v \\ 0 & 1 & w \\ 0 & 0 & h^{-1} \end{pmatrix} \; : \; u, v, w \in \mathbb{F}_p, h \in \mathbb{F}_p^* \right\}$$

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is an evolving group.

By direct computation, one gets $G = G_{\mathrm{H}} \rtimes \mathbb{F}_p^* = S_p \rtimes T_p$; it follows that G has a Sylow family and that G has the correction property.

The subgroup

$$W = \left\{ \begin{pmatrix} h & u & v \\ 0 & 1 & 0 \\ 0 & 0 & h^{-1} \end{pmatrix} \; : \; u, v \in \mathbb{F}_p, h \in \mathbb{F}_p^* \right\}$$

of G is not evolving.

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Indeed, if we write $W = W_{\mathrm{H}} \rtimes \mathbb{F}_p^*$, then $\mathbb{F}_p^*$ does not act on W_{H} by scalar multiplications.

Thank you.