Intense automorphisms of finite groups

Mima Stanojkovski – this is part of my PhD thesis, supervised by Prof. Hendrik Lenstra (Universiteit Leiden)

Universität Bielefeld

mstanojk@math.uni-bielefeld.de

Intense automorphisms

Let G be a group. An automorphism α of G is *intense* if for all subgroups H of G there exists $g \in G$ such that $\alpha(H) = gHg^{-1}$. Denote by $\mathrm{Int}(G)$ the collection of intense automorphisms of G; then $\mathrm{Int}(G) \lhd \mathrm{Aut}(G)$.

Examples:

1. Inner automorphisms are intense.
2. If V is a vector space over the finite field $\mathbb{F}_p$ of p elements, then the intense automorphisms of V are the scalar multiplications by elements of $\mathbb{F}_p^*$.

Equivalence relation: Let G, G' be groups and let α, β be intense automorphisms respectively of G and G'. The pairs (G, α) and (G', β) are *equivalent* if there exists an isomorphism $\sigma : G \to G'$ such that $\beta\sigma = \sigma\alpha$.

The general setting

Let G be a finite group. A lot can be said about the structure of G once the structure of $\mathrm{Aut}(G)$ is known. Besides, in some cases, very few assumptions on $\mathrm{Aut}(G)$ can lead to very strong limitations to the shape of G.

Intense automorphisms are a generalization of power automorphisms and, in some sense, they resemble class-preserving automorphisms. If G is a non-abelian p-group then both power and class-preserving automorphisms have order equal to a power of p, but the same need not hold for the elements of $\mathrm{Int}(G)$. We will explore this last situation extensively and see how intense automorphisms give rise to a surprisingly rich theory.

The case of groups of prime power order

Let p be a prime number and let G be a finite p-group. Then

$$\mathrm{Int}(G) = P_G \rtimes C_G,$$

where P_G is the unique Sylow p-subgroup of $\mathrm{Int}(G)$ and C_G is a cyclic group of order dividing $p-1$. The *intensity* of G is $\mathrm{int}(G) = \#C_G$.

Goal: Classifying finite p-groups G whose group of intense automorphisms $\mathrm{Int}(G)$ is not itself a p-group. In other words, we want to know for which groups G one has $\mathrm{int}(G) > 1$. *As this can never happen for 2-groups, we restrict ourselves to working with odd primes.*

Strategy: Let $\mathcal{T}_p$ be the collection of equivalence classes of pairs (G, α) such that G is a finite p-group and α is conjugate to a *non-trivial* element of C_G. For all $c \in \mathbb{Z}_{\geq 0}$, define

$$\mathcal{T}_p[c] = \{\, [(G,\alpha)] \in \mathcal{T}_p \;:\; G \text{ has class } c \,\}$$

and note that the collection $\{\,\mathcal{T}_p[c]\,\}_{c\geq 0}$ is a partition of $\mathcal{T}_p$.

Small nilpotency classes

Let p be an odd prime number. The following hold.

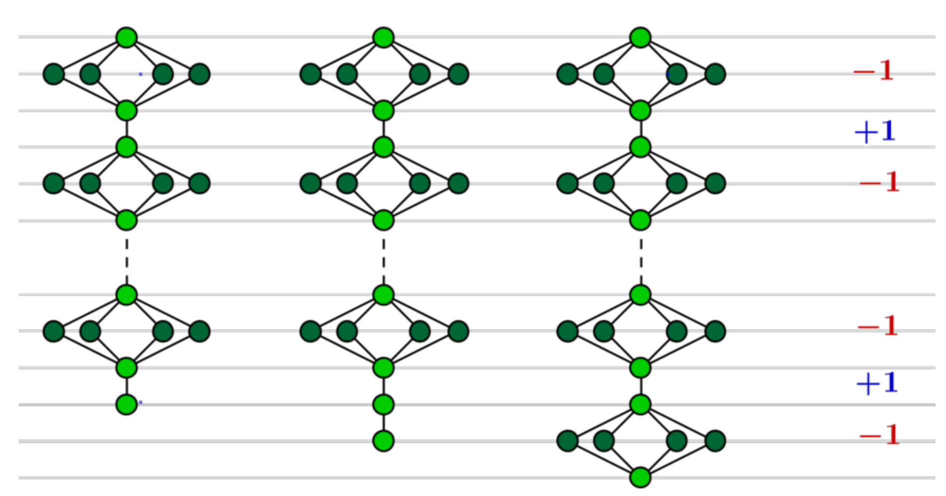
1. One has
$$\mathcal{T}_p[1] = \{\, [(G,\alpha)] \;:\; G \neq 1 \text{ abelian},\ \alpha \in \omega(\mathbb{F}_p^*) \setminus \{1\} \,\},$$
where $\omega : \mathbb{F}_p^* \to \mathbb{Z}_p^*$ is the Teichmüller character.
2. One has
$$\mathcal{T}_p[2] = \{\, [(\mathrm{ES}_p(n), \alpha_\lambda)] \;:\; n \in \mathbb{Z}_{\geq 1}, \lambda \in \mathbb{F}_p^* \setminus \{1\} \,\},$$
where $\mathrm{ES}_p(n)$ is extraspecial, of order p^{2n+1} and exponent p, and α_λ is a lift of λ-th powering on $\mathrm{ES}_p(n)/\Phi(\mathrm{ES}_p(n))$.

Note: If G is a finite p-group of class at most 2, then $\mathrm{int}(G)$ is either 1 or $p-1$. Moreover, $\mathcal{T}_p[1]$ and $\mathcal{T}_p[2]$ are both infinite.

Higher nilpotency classes

Let p be an odd prime number, $c \geq 3$, and $[(G,\alpha)] \in \mathcal{T}_p[c]$. Then the following hold.

1. One has $|\alpha| = 2$ and $\mathrm{int}(G) = 2$.
2. The lower central series and p-central series of G coincide.
3. The map α induces the inversion map on $G/\Phi(G)$.
4. The group G is thin, with one of the following diagrams.



Theorem

Let p be odd and let $c \in \mathbb{Z}_{>0}$. Then the following hold.

1. If $c \geq 3$, then $\mathcal{T}_p[c]$ is finite.
2. $\mathcal{T}_p[c] = \emptyset \iff p = 3$ and $c \geq 5$.
3. The set $\mathcal{T}_3[4]$ has exactly one element.
4. If $p > 3$, then $\#\varprojlim_c \mathcal{T}_p[c] = 1$.

If $\varprojlim_c \mathcal{T}_p[c] = \{[(G^{(c)}, \alpha^{(c)})]\}_{c>0}$, we want to determine the pro-p-group $G_{\lim} = \varprojlim_c G^{(c)}$ and the automorphism $\alpha_{\lim}$ of $G_{\lim}$ that is induced by the automorphisms $\alpha^{(c)}$.

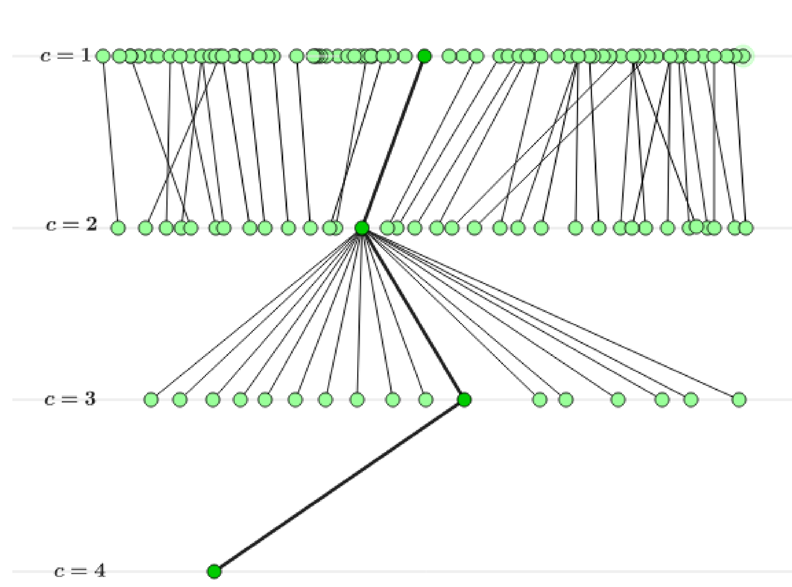
INTENSE PROJECTIVE SYSTEM

There is a well-defined sequence of sets

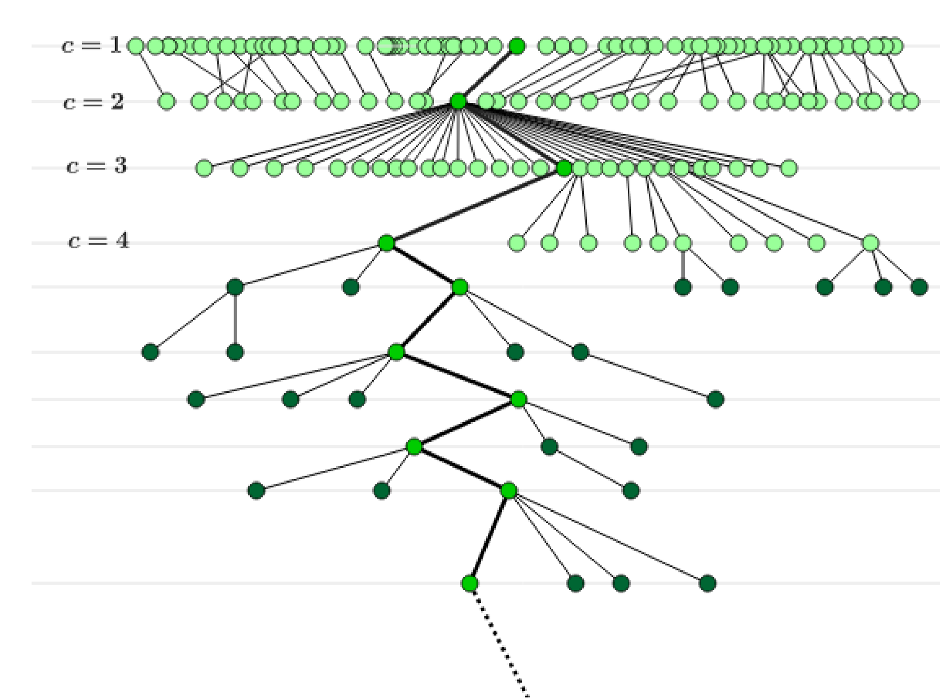
$$\ldots \longrightarrow \mathcal{T}_p[c+1] \xrightarrow{\pi_{c+1}} \mathcal{T}_p[c] \xrightarrow{\pi_c} \mathcal{T}_p[c-1] \longrightarrow \quad \ldots \quad \longrightarrow \mathcal{T}_p[1]$$

where, for all c, the map π_c is defined by $\pi_c : [(G,\alpha)] \mapsto [(G/\gamma_c(G), \overline{\alpha})]$.
The sequence $(\gamma_i(G))_{i\geq 1}$ denotes the lower central series of G and $\overline{\alpha}$ is the map induced by α on $G/\gamma_c(G)$.

The projective system for $p = 3$



The projective system for $p \geq 5$



A maximal class example

Let $k = \mathbb{F}_3[\epsilon]$ of cardinality 9 with $\epsilon^2 = 0$ and set

$$\mathrm{A}_3 = k + k\mathrm{i} + k\mathrm{j} + k\mathrm{ij},$$

where i, j satisfy $\mathrm{i}^2 = \mathrm{j}^2 = \epsilon$, and $\mathrm{ji} = -\mathrm{ij}$. The quaternion algebra A_3 is local, with maximal ideal $\mathfrak{m} = \mathrm{A}_3\mathrm{i} + \mathrm{A}_3\mathrm{j}$ and canonical anti-homomorphism

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \;\mapsto\; \overline{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Define $G_{\max} = \{a \in 1 + \mathfrak{m} \;:\; a\overline{a} = 1\}$ and let the automorphism $\alpha_{\max} : G_{\max} \to G_{\max}$ be defined by

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \;\mapsto\; \alpha_{\max}(a) = s - t\mathrm{i} - u\mathrm{j} + v\mathrm{ij}.$$

Fact: $\mathcal{T}_3[4] = \{[(\,G_{\max}, \alpha_{\max}\,)]\}$.

Another construction

Let J_3 denote the third Janko group and let $3.\mathrm{J}_3$ denote its Schur cover. Let S be a Sylow 3-subgroup of $3.\mathrm{J}_3$ and denote by N its normalizer. Let x be an element of order 2 in N and let $\iota_x : S \to S$ be conjugation under x.

Fact: $\mathcal{T}_3[4] = \{[(S, \iota_x)]\}$.

Thanks to: Derek Holt and Frieder Ladisch for this characterization.

A profinite example

Let $p > 3$ be a prime number and let $t \in \mathbb{Z}_p$ satisfy $\left(\frac{t}{p}\right) = -1$. Set $\mathrm{A}_p = \mathbb{Z}_p + \mathbb{Z}_p\mathrm{i} + \mathbb{Z}_p\mathrm{j} + \mathbb{Z}_p\mathrm{ij}$ with defining relations $\mathrm{i}^2 = t$, $\mathrm{j}^2 = p$, and $\mathrm{ji} = -\mathrm{ij}$. Then A_p is a non-commutative local ring such that $\mathrm{A}_p/\mathrm{j}\mathrm{A}_p \cong \mathbb{F}_{p^2}$. The involution $\overline{\cdot} : \mathrm{A}_p \to \mathrm{A}_p$ is defined by

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \;\mapsto\; \overline{a} = s - t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Let $\mathrm{SL}(p) = \{a \in 1 + \mathrm{j}\mathrm{A}_p \;:\; a\overline{a} = 1\}$ and let α_p be the automorphism of $\mathrm{SL}(p)$ that is defined by

$$a = s + t\mathrm{i} + u\mathrm{j} + v\mathrm{ij} \;\mapsto\; \alpha_p(a) = s + t\mathrm{i} - u\mathrm{j} - v\mathrm{ij}.$$

Theorem

The group $\mathrm{SL}(p)$ is a pro-p-group and α_p is topologically intense, i.e. for any closed subgroup H of $\mathrm{SL}(p)$ there exists $g \in \mathrm{SL}(p)$ such that $\alpha_p(H) = gHg^{-1}$. Moreover, one has

$$(\,\mathrm{SL}(p), \alpha_p) \cong (\,G_{\lim}, \alpha_{\lim}\,).$$