EVOLVING GROUPS AND INTENSITY

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What is an evolving group?

An *evolving group* is a finite group G in which, for every prime number p and for every p-subgroup I, there is a subgroup J such that the following hold.

- $I \subseteq J$
- $|J : I|$ is coprime to p
- $|G : J|$ is a p-power.

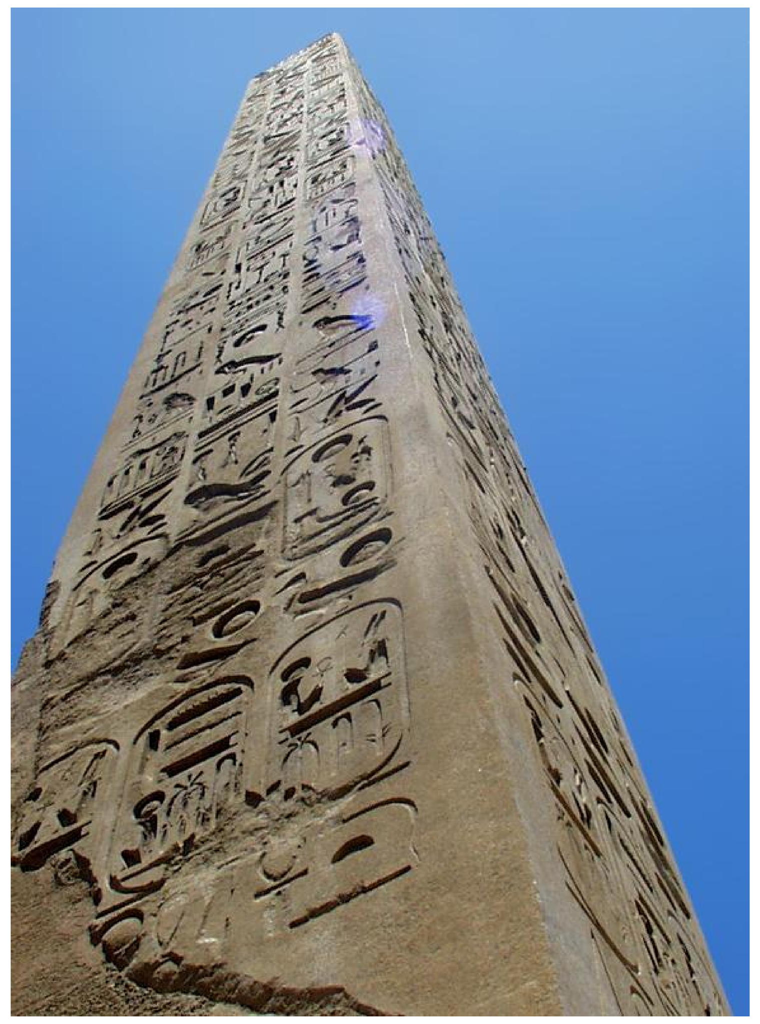
Examples of evolving groups are given by nilpotent groups or groups in which every Sylow subgroup is cyclic.

Motivation and goals

Let G be a finite group. Then the following are equivalent.

1. For every G-module M, every integer q, and every $c \in \mathrm{H}^q(G, M)$, the minimum of the set $\{|G : H| \mid H \leq G \text{ with } c \in \ker \mathrm{Res}^G_H\}$ coincides with its greatest common divisor.
2. The group G is an evolving group.

The starting point of the project was the curiosity of understanding the properties of the groups satisfying condition (1), which is inspired by a phenomenon occurring in Galois cohomology. Once we had translated the purely cohomological requirement into group theoretic terms, we began investigating the internal structure of these groups and the interactions between their Sylow subgroups.



Intense automorphisms

Let G be a finite group. We say that $\alpha \in \mathrm{Aut}(G)$ is an *intense automorphism* if for every subgroup H of G there exists $g \in G$ such that $\alpha(H) = gHg^{-1}$; we denote by $\mathrm{Int}(G)$ the collection of all intense automorphisms of G. It is easy to see that $\mathrm{Inn}(G) \leq \mathrm{Int}(G) \lhd \mathrm{Aut}(G)$.

Properties of evolving groups

Let G be an evolving group. Then the following are satisfied.

1. If $N \lhd G$, then both N and G/N are evolving.
2. G is supersolvable.
3. G equals the semidirect product of two nilpotent subgroups of coprime orders.

Example

Let $p > 2$ be prime and let $G = \mathrm{Heis}(\mathbb{F}_p) \rtimes_\phi \mathbb{F}_p^*$, where

- G can be represented in $\mathrm{GL}(3, \mathbb{F}_p)$ as the subgroup of matrices of the form $\begin{pmatrix} h & u & v \\ 0 & 1 & w \\ 0 & 0 & h^{-1} \end{pmatrix}$ with $u, v, w \in \mathbb{F}_p, h \in \mathbb{F}_p^*$.
- $\mathrm{Heis}(\mathbb{F}_p)$ is the Heisenberg group modulo p in $\mathrm{GL}(3, \mathbb{F}_p) \rightsquigarrow$ in G take $h = 1$.
- ρ is the representation of $\mathbb{F}_p^*$ in $\mathrm{GL}(3, \mathbb{F}_p)$ that sends $h \in \mathbb{F}_p^*$ to the diagonal matrix with entries $(h, 1, h^{-1})$.
- $\phi : \mathbb{F}_p^* \to \mathrm{Aut}(\mathrm{Heis}(\mathbb{F}_p))$ is obtained by composing conjugation in $\mathrm{GL}(3, \mathbb{F}_p)$ with ρ.

The following hold.

- G is an evolving group.
- G has a non-evolving subgroup W that is supersolvable and can be written as the semidirect product of two nilpotent groups of coprime orders.
- $\mathbb{F}_p^*$ acts faithfully on $\mathrm{Heis}(\mathbb{F}_p)$ and so $\phi(\mathbb{F}_p^*)$ is cyclic of order $p - 1$.
- $\phi(\mathbb{F}_p^*) \subseteq \mathrm{Int}(\mathrm{Heis}(\mathbb{F}_p))$.

Classification

Classifying evolving groups reduces to understanding the group of intense automorphisms of a finite nilpotent group, as we can see from the following result.

Recipe for evolving groups

Let G be a finite group. The following are equivalent.

1. G is evolving.
2. There are nilpotent groups N and T of coprime orders and a group homomorphism $\phi : T \to \mathrm{Int}(N)$ such that $G = N \rtimes_\phi T$.

An equivalent condition

Assume N is a group and Q is a subgroup of $\mathrm{Aut}(N)$ that has order coprime with the order of N. By Glauberman's lemma, we have that for all $\alpha \in Q$ the following are equivalent.

1. $\alpha \in \mathrm{Int}(N)$.
2. For every subgroup H of N, the subgroup H has an α-stable N-conjugate.

The intensity of p-groups

Let p be a prime number and let G be a finite p-group. Then the following hold.

1. $\mathrm{Int}(G)$ has a unique Sylow p-subgroup $\mathrm{Int}(G)_p$.
2. The action of $\mathrm{Int}(G)$ on the Frattini quotient of G induces an injective homomorphism $\mathrm{Int}(G)/\,\mathrm{Int}(G)_p \longrightarrow \mathbb{F}_p^*$.

We call the index of $\mathrm{Int}(G)_p$ in $\mathrm{Int}(G)$ the *intensity* of G and we denote it by $\mathrm{int}(G)$. Observe that if G is a 2-group, its intensity always equals 1.

Question: Can we determine the intensity of every p-group?

Partial result: Let $p > 2$ be a prime and let G be a p-group of nilpotency class c. Then the following hold.

0. If $c = 0$, then $\mathrm{int}(G) = 1$.
1. If $c = 1$, then $\mathrm{int}(G) = p - 1$.
2. If $c = 2$, then exactly one of the following holds.
   - $\mathrm{int}(G) = 1$.
   - G is extraspecial of exponent p and $\mathrm{int}(G) = p - 1$.
3. Groups of class at least 3 have intensity at most 2.
4. Assume $c > 2$. Then the number of isomorphism types of p-groups of class c and intensity 2 is finite.

Future goals

Our major goal is understanding all p-groups of intensity 2 and class at least 3.
Are there for any p infinitely many such groups?

"You look stiff and hard enough," said I, "but where are your hieroglyphics? That's the test of a true obelisk – the hieroglyphics."

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