Subspaces fixed by a nilpotent matrix

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- Y. El Maazouz, M. A. Hahn, G. Nebe, M. Stanojkovski, B. Sturmfels, *Orders and Polytropes: Matrix Algebras from Valuations*, Beitr. Algebra Geom. 63, 515-531 (2022).
- Y. El Maazouz, G. Nebe, M Stanojkovski, *Bolytrope orders*, Int. J. Number Theory (2022).
- M. Stanojkovski, *Submodule codes as spherical codes in buildings*, arXiv 2202.13370.
- M. A. Hahn, G. Nebe, M. Stanojkovski, B. Sturmfels, Subspaces fixed by a nilpotent matrix, arXiv 2207.00802.

The linear spaces that are fixed by a given nilpotent $n \times n$ matrix form a subvariety of the Grassmannian. We classify these varieties for small n. Mutiah, Weekes and Yacobi conjectured that their radical ideals are generated by certain linear forms known as shuffle equations. We prove this conjecture for $n \leq 7$, and we disprove it for $n = 8$. The question remains open for nilpotent matrices arising from the affine Grassmannian.

D. Muthiah, A. Weekes and O. Yacobi: *The equations defining affine Grassmannians in type A and a conjecture of Kreiman, Lakshmibai, Magyar, and Weyman*, International Mathematics Research Notices **3** (2022) 1922-1972.

For a field K, the Grassmannian $\mathrm{Gr}(\ell, n)$ parametrizes the ℓ-dimensional vector subspaces of K^n.

Each $L \in \mathrm{Gr}(\ell, n)$ can be represented as the row space of a $(\ell \times n)$-matrix $\mathbf{L}$.

Alternatively, L can be represented via its Plücker coordinates, i.e. via its image under the Plücker embedding $\mathrm{Gr}(\ell, n) \to \mathbb{P}^{\binom{n}{\ell}-1}$ where $\mathbf{L}$ is mapped to the vector of its maximal minors $p_{i_1 i_2 \ldots i_\ell}$.

Remark. The Grassmannian $\mathrm{Gr}(\ell, n)$ is Zariski closed and irreducible. Its dimension is $(n-\ell)\ell$ and its homogeneous prime ideal has a natural Gröbner basis of quadrics, known as the Plücker quadrics.

 ## An example for $\ell = 2$ and $n = 4$

Take $\ell = 2$ and $n = 4$.

The Grassmannian $\mathrm{Gr}(2,4)$ is the image of the map

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \mapsto (af - be : ag - ce : bg - cf : ah - de : bh - df : ch - dg) \in \mathbb{P}^5$$

and parametrizes the 2-dimensional subspaces of K^4.

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Writing $(p_{12} : p_{13} : p_{23} : p_{14} : p_{24} : p_{34})$ for the coordinates in $\mathbb{P}^5$, its prime ideal is generated by the Plücker quadric

$$p_{23}p_{14} - p_{13}p_{24} + p_{12}p_{34}.$$

Its dimension is 4 and thus $\mathrm{Gr}(2,4)$ is a hypersurface in $\mathbb{P}^5$.

The T-Grassmannian

Let $T \in K^{n\times n}$ be nilpotent, i.e. $T^n = 0$.

The T-Grassmannian $\mathrm{Gr}(\ell, n)^T$ is the subvariety of $\mathrm{Gr}(\ell, n)$ that parametrizes all L in $\mathrm{Gr}(\ell, n)$ with $LT \subseteq L$.

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Example. If $T = 0$, then $\mathrm{Gr}(\ell, n)^T = \mathrm{Gr}(\ell, n)$. If $\ell = 1$, then $\mathrm{Gr}(\ell, n)^T$ is the same as $\ker T$.

We are interested in the homogeneous radical ideal of $\mathrm{Gr}(\ell, n)^T$ in the Plücker coordinates $p_{i_1 \ldots i_\ell}$.

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Theorem. (Mutiah, Weekes, Yacobi) The variety $\mathrm{Gr}(\ell, n)^T$ is the intersection of the Grassmannian $\mathrm{Gr}(\ell, n)$ with a linear subspace in $\mathbb{P}^{\binom{n}{\ell}-1}$. That linear subspace is defined by the *shuffle equations*.

Conjecture. (Mutiah, Weekes, Yacobi) The shuffle equations plus the Plücker quadrics generate the radical ideal of $\mathrm{Gr}(\ell, n)^T$.

Another example for $\ell = 2$ and $n = 4$

Fix $\epsilon \in K$ and let

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The ideal of $\mathrm{Gr}(2,4)^T$ is

$(\epsilon \neq 0)$ prime and equal to

$$\langle p_{13},\, p_{14}+\epsilon\, p_{23},\, p_{12}p_{34}-\epsilon p_{23}^2 \rangle = \langle p_{13},\, p_{14}+\epsilon\, p_{23} \rangle + \text{ideal of } \mathrm{Gr}(2,4),$$

$(\epsilon = 0)$ radical (but not prime!) and equal to

$$\langle p_{13}\,,\, p_{14}\,,\, p_{12}p_{34} \rangle = \langle p_{13}, p_{14}, p_{34} \rangle \cap \langle p_{12}, p_{13}, p_{14} \rangle.$$

So $\mathrm{Gr}(2,4)^T$ can be singular or reducible. For any ϵ, its radical ideal is generated by two linear forms plus the Plücker quadric $p_{12}p_{34} - p_{13}p_{23} + p_{14}p_{23}$.

Assume that T is in Jordan canonical form. The necessary change of basis in K^n works over K as all eigenvalues of T are 0!

Write $T = T_\lambda$ with $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s)$ a partition of n.

Example. The operator from the previous slide is in JNF if $\epsilon = 0$ or $\epsilon = 1$, in which case $\lambda = (2,1,1)$ resp. $\lambda = (2,2)$.

Example. If $\lambda = (4,2,2)$, then

$$T = T_\lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If $z \in K$, then $\mathrm{Id}_n + zT$ is an automorphism of K^n. Moreover,

$$LT \subseteq L \iff L(\mathrm{Id}_n + zT) = L \text{ for all } z.$$

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Writing $\wedge_\ell(\mathrm{Id}_n + zT) = \wedge_\ell \mathrm{Id}_n + \sum_{i=1}^{\ell} [\wedge_\ell(\mathrm{Id}_n + zT)]_i z^i$:

$$LT \subseteq L \iff P \cdot [\, \wedge_\ell(\mathrm{Id}_n + zT)\,]_i = 0 \text{ for } i = 1, 2, \dots, \ell.$$

This is a finite collection of linear forms in the $\binom{n}{\ell}$ Plücker coordinates $p_{i_1 i_2 \cdots i_\ell}$. These are the shuffle equations of T.

Let T be as before and $P = (p_{12}, p_{13}, p_{23}, p_{14}, p_{24}, p_{34})$. Then

$$\wedge_2(\mathrm{Id}_4 + zT) \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & z & \epsilon z & \epsilon z^2 & 0 \\ 0 & 0 & 1 & 0 & \epsilon z & 0 \\ 0 & 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

and so

$$P \cdot [\,\wedge_2(\mathrm{Id}_4 + zT)\,]_1 = (0, 0, p_{13}, \epsilon p_{13}, \epsilon p_{23} + p_{14}, 0),$$
$$P \cdot [\,\wedge_2(\mathrm{Id}_4 + zT)\,]_2 = (\,0,\ 0,\ 0,\ 0,\ \epsilon p_{13},\ 0\,).$$

The coordinates are the shuffle equations.

Theorem. (Mutiah, Weekes, Yacobi) The variety $\mathrm{Gr}(\ell, n)^T$ is the intersection of the Grassmannian $\mathrm{Gr}(\ell, n)$ with a linear subspace in $\mathbb{P}^{\binom{n}{\ell}-1}$. That linear subspace is defined by the shuffle equations.

Remark.

- MWY conjectured that the ideal generated by the Plücker quadrics and the shuffle equations is radical.
- We use computer algebra systems (`Macaulay2`) to settle this conjecture for $n \leq 8$.

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Theorem. The varieties $\mathrm{Gr}(\ell, n)^T$ and $\mathrm{Gr}(n-\ell, n)^T$ coincide after a linear change of coordinates in the ambient space $\mathbb{P}^{\binom{n}{\ell}-1}$. Under this coordinate change, which depends on T, the shuffle equations coincide.

Theorem. Fix $1 \le \ell < n \le 7$ and let T be any nilpotent $n \times n$ matrix. Then the shuffle equations generate the radical ideal of the fixed point locus $\mathrm{Gr}(\ell, n)^T$. The same does not hold for $n = 8$: there is a unique partition, namely $\lambda = (4, 2, 2)$, and a unique dimension, namely $\ell = 4$, such that the radical ideal of $\mathrm{Gr}(\ell, n)^{T_\lambda}$ is not generated by the shuffle equations.

For each (λ, ℓ), we write $[\sigma, \delta, \gamma]^\kappa$ meaning:

- σ is the number of linearly independent shuffle equations,
- δ and γ are the dimension and degree of $\mathrm{Gr}(\ell, n)^{T_\lambda}$ in $\mathbb{P}^{\binom{n}{\ell}-1}$,
- κ is the number of irreducible components of $\mathrm{Gr}(\ell, n)^T$.

- $n = 2$: here $\ell = 1$ and so we reduce to $\ker T \subseteq K^n$
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- $n = 3$: here duality allows us to reduce to $\ell = 1$
- for $4 \leq n \leq 8$, we compute:

λ	$\ell = 1$	$\ell = 2$
(1,1,1,1)	[0,3,1]	[0,4,2]
(2,1,1)	[1,2,1]	$[2,2,2]^2$
(2,2)	[2,1,1]	[2,2,2]
(3,1)	[2,1,1]	[4,1,1]
(4)	[3,0,1]	[5,0,1]

λ	$\ell = 1$	$\ell = 2$
(1,1,1,1,1)	[0,4,1]	[0,6,5]
(2,1,1,1)	[1,3,1]	$[3,4,2]^2$
(2,2,1)	[2,2,1]	[4,3,3]
(3,1,1)	[2,2,1]	$[6,2,2]^2$
(3,2)	[3,1,1]	[6,2,2]
(4,1)	[3,1,1]	[8,1,1]
(5)	[4,0,1]	[9,0,1]

Table 1: Fixed point loci $\mathrm{Gr}(\ell, n)^T$ for $n = 4$ and $n = 5$.

λ	$\ell=1$	$\ell=2$	$\ell=3$
(1,1,1,1,1,1)	[0,5,1]	[0,8,14]	[0,9,42]
(2,1,1,1,1)	[1,4,1]	$[4,6,5]^2$	$[6,6,10]^2$
(2,2,1,1)	[2,3,1]	$[6,4,6]^2$	[8,5,10]
(3,1,1,1)	[2,3,1]	$[8,4,2]^2$	$[12,4,2]^3$
(2,2,2)	[3,2,1]	[6,4,6]	[11,4,6]
(3,2,1)	[3,2,1]	[9,3,3]	$[12,3,6]^2$
(4,1,1)	[3,2,1]	$[11,2,2]^2$	$[16,2,2]^2$
(3,3)	[4,1,1]	[11,2,2]	[12,3,6]
(4,2)	[4,1,1]	[11,2,2]	[16,2,2]
(5,1)	[4,1,1]	[13,1,1]	[18,1,1]
(6)	[5,0,1]	[14,0,1]	[19,0,1]

λ	$\ell=1$	$\ell=2$	$\ell=3$
(1,1,1,1,1,1,1)	[0,6,1]	[35,10,42]	[140,12,462]
(2,1,1,1,1,1)	[1,5,1]	$[5,8,14]^2$	$[10,9,42]^2$
(2,2,1,1,1)	[2,4,1]	$[8,6,5]^2$	$[14,7,35]^2$
(3,1,1,1,1)	[2,4,1]	$[10,6,5]^2$	$[20,6,10]^3$
(2,2,2,1)	[3,3,1]	[9,5,10]	[17,6,30]
(3,2,1,1)	[3,3,1]	$[12,4,6]^2$	$[21,5,10]^2$
(4,1,1,1)	[3,3,1]	$[14,4,2]^2$	$[27,4,2]^3$
(3,2,2)	[4,2,1]	[12,4,6]	$[23,4,12]^2$
(3,3,1)	[4,2,1]	[15,3,3]	[23,4,12]
(4,2,1)	[4,2,1]	[15,3,3]	$[27,3,6]^2$
(5,1,1)	[4,2,1]	$[17,2,2]^2$	$[31,2,2]^2$
(4,3)	[5,1,1]	[17,2,2]	[27,3,6]
(5,2)	[5,1,1]	[17,2,2]	[31,2,2]
(6,1)	[5,1,1]	[19,1,1]	[33,1,1]
(7)	[6,0,1]	[20,0,1]	[34,0,1]

Table 2: Fixed point loci $\mathrm{Gr}(\ell,n)^T$ for $n=6$ and $n=7$.

λ	$\ell = 2$	$\ell = 3$	$\ell = 4$
(1,1,1,1,1,1,1,1)	[0,12,132]	[420,15,6006]	[721,16,24024]
(2,1,1,1,1,1,1)	$[6,10,42]^2$	$[15,12,462]^2$	$[20,12,924]^2$
(2,2,1,1,1,1)	$[10,8,14]^2$	$[22,9,168]^2$	$[28,10,420]^3$
(3,1,1,1,1,1)	$[12,8,14]^2$	$[30,9,42]^3$	$[40,9,42]^3$
(2,2,2,1,1)	$[12,6,20]^2$	[26,8,140]	$[34,8,280]^2$
(3,2,1,1,1)	$[15,6,5]^2$	$[33,7,35]^3$	$[42,7,70]^2$
(4,1,1,1,1)	$[17,6,5]^2$	$[41,6,10]^3$	$[54,6,10]^4$
(2,2,2,2)	[12,6,20]	[32,7,70]	[34,8,280]
(3,2,2,1)	[16,5,10]	$[35,6,30]^2$	$[46,6,60]^2$
(3,3,1,1)	$[19,4,6]^2$	$[38,5,30]^2$	[46,6,60]
(4,2,1,1)	$[19,4,6]^2$	$[42,5,10]^2$	$[54,5,10]^3$
(5,1,1,1)	$[21,4,2]^2$	$[48,4,2]^3$	$[62,4,2]^3$
(3,3,2)	[19,4,6]	[38,5,30]	[52,5,30]
(4,2,2)	[19,4,6]	$[44,4,12]^2$	$\mathbf{[54,4,24]^3}$
(4,3,1)	[22,3,3]	[44,4,12]	$[54,4,24]^2$
(5,2,1)	[22,3,3]	$[48,3,6]^2$	$[62,3,6]^2$
(6,1,1)	$[24,2,2]^2$	$[52,2,2]^2$	$[66,2,2]^2$
(4,4)	[24,2,2]	[48,3,6]	[54,4,24]
(5,3)	[24,2,2]	[48,3,6]	[62,3,6]
(6,2)	[24,2,2]	[52,2,2]	[66,2,2]
(7,1)	[26,1,1]	[54,1,1]	[68,1,1]
(8)	[27,0,1]	[55,0,1]	[69,0,1]

Table 3: Fixed point loci $\mathrm{Gr}(\ell, n)^T$ for $n = 8$.

A rectangular partition is $\lambda = (\underbrace{r, \ldots, r}_{d \text{ times}})$ so $n = dr$.

Theorem. If λ is a rectangular partition then the variety $\mathrm{Gr}(\ell, n)^{T_\lambda}$ is irreducible.

Conjecture. The MWY-conjecture holds for rectangular partitions λ. In other words, for rectangular partitions, the shuffle equations plus Plücker quadrics generate a prime ideal.

Remark. Conjecture true for $n \leq 8$ where the rectangular partitions are $(2, 2)$, $(2, 2, 2)$, $(2, 2, 2, 2)$, $(3, 3)$, $(4, 4)$. Also true for $\lambda = (1, \ldots, 1)$ and $\lambda = (n)$. More supporting evidence: $n = 9$ and some cases for $n = 10$ or $n = 12$.

Let $\mathbb{K} = K(\!(t)\!)$ be the field of Laurent series with coefficients in K. Its valuation ring is $\mathcal{O}_{\mathbb{K}} = K[[t]]$ and its residue field is K.

A lattice L is an $\mathcal{O}_{\mathbb{K}}$-submodule of $\mathbb{K}^d$ of maximal rank d. It can be represented by a $d \times d$ matrix with entries in $\mathbb{K}$.

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To obtain finite-dimensional varieties we can truncate:

$$\mathcal{B}_r = \{L \text{ lattice } : t^r \mathcal{O}_{\mathbb{K}}^d \subseteq L \subseteq \mathcal{O}_{\mathbb{K}}^d\}.$$

Modulo *equivalence of lattices* $\mathcal{B}_r$ is the ball of radius r around $\mathcal{O}_{\mathbb{K}}^d$ in the Bruhat-Tits building for $\mathrm{GL}_d(\mathbb{K})$.

Both $t^r\mathcal{O}_{\mathbb{K}}^d$ and $\mathcal{O}_{\mathbb{K}}^d$ are infinite-dimensional vector spaces over K, but their quotient $K^n = \mathcal{O}_{\mathbb{K}}^d \,/\, t^r\mathcal{O}_{\mathbb{K}}^d$ has dimension $n = dr$.

Moreover, every lattice $L \in \mathcal{B}_r$ is determined by its image in K^n, i.e. there is some ℓ such that $L \in \mathrm{Gr}(\ell, n)$ and $tL \subseteq L$.

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Moreover, every lattice $L \in \mathcal{B}_r$ is determined by its image in K^n, i.e. there is some ℓ such that $L \in \mathrm{Gr}(\ell, n)$ and $tL \subseteq L$.

If $e_1, e_2, \ldots, e_d$ is the standard basis of K^d, then

$$e_1, te_1, \ldots, t^{r-1}e_1,\ e_2, te_2, \ldots, t^{r-1}e_2,\ \ldots\ldots,\ e_d, te_d, \ldots, t^{r-1}e_d$$

is a basis for K^n and, wrt to this basis, multiplication by t is given by T_λ for $\lambda = (r, \ldots, r)$.

Consequence. $\mathcal{B}_r = \bigcup_{\ell=0}^{dr} \mathrm{Gr}(\ell, n)^{T_\lambda}$ where $\lambda = (r, r, \ldots, r)$.

Our next goal is to study Λ-Grassmannians, where Λ is an *order* in $K^{d\times d}$. These are unions of Grassmannians of lattices that are stable under the elements of Λ.

Example. If Λ is a *ball order*, then the Λ-Grassmannian is a ball in the Bruhat-Tits building.

Next candidates on the list are *bolytrope orders*...

This is not ⟩ the end of the story

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